Riccati-Based Feedback Stabilization of Flow Problems

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1. SUMMARY

Our goal is to derive and investigate effective numerical algorithms for the stabilization of flow problems governed by the Navier Stokes equations around an unstable steady state solution w.

Linearizing the Navier Stokes equations for the difference of w and the instationary solution z, we obtain the so-called Oseen equations for $y = z - w$. We want to minimize y by a boundary feedback control. In particular, the control will have a non-zero normal component.

Reformulating the equations allows us to apply Riccati theory and derive an algebraic Riccati equation (ARE) from which we can calculate an optimal control u.

For the numerical solution, we discretize the system by a Galerkin finite element method. For the solution of the ARE, we use a Newton-based algorithm exploiting the structure of the discretized operators.

2. THEORETICAL APPROACH

For the Oseen equations with a given stationary solution w and the instationary solution y that we want to stabilize, we can formulate an optimal boundary control problem as follows:

$$\inf\{J(y,u) :\ (y,u) \text{ fulfill } (1),\ u \in V^{0,0}(\Sigma)\},$$

$$J(y,u) = \frac{1}{2}\int_0^T \int_\Omega |y|^2 \, dx\, dt + \frac{1}{2}\int_0^T |u|^2_{V^0(\Gamma)} dt,$$

$$\begin{aligned}
&\partial_t y - \frac{1}{\mathrm{Re}}\Delta y + (y\cdot\nabla)w + (w\cdot\nabla)y + \nabla p = 0,\\
&\operatorname{div} y = 0 \quad \text{in} \quad Q = \Omega \times (0,T),\\
&y = Mu \quad \text{on} \quad \Sigma = \Gamma \times (0,T),\\
&y(0) = \zeta \quad \text{in} \quad \Omega.
\end{aligned} \tag{1}$$

M restricts the control u to a part of the boundary, $T > 0$ can be finite or infinite, the divergence free spaces

$$\begin{aligned}
V^0(\Gamma) = \{u \in L^2(\Gamma) :\ &\operatorname{div} u = 0 \text{ in } \Omega,\\
&< u\cdot n, 1 >_{H^{-1/2}(\Gamma),H^{1/2}(\Gamma)} = 0\},
\end{aligned}$$

$$V^{0,0}(\Sigma) = L^2(0,T; V^0(\Gamma))$$

allow controls with a nonzero normal component.

Using the orthogonal Helmholtz projection

$$P :\ L^2(\Omega) \to V^0_n(\Omega),$$

$$V^0_n(\Omega) = \{u \in L^2(\Omega) :\ \operatorname{div} u = 0 \text{ in } \Omega,\ u\cdot n = 0\},$$

the Dirichlet operator D_A defined by $D_A w = v$ iff

$$\begin{aligned}
&\lambda v - \frac{1}{\mathrm{Re}}\Delta v + (w\cdot\nabla)v + (v\cdot\nabla)w + \nabla\pi = 0,\\
&\operatorname{div} v = 0 \quad \text{in} \quad \Omega,\\
&v = w \quad \text{on} \quad \Gamma,
\end{aligned}$$

and the boundary projectors

$$\gamma_n u = (u\cdot n)n, \quad \gamma_\tau u = u - \gamma_n u,$$

in (5) Raymond derives the equivalent problem

$$\inf\{I(y,u) :\ (y,u) \text{ fulfill } (2),\ u \in V^{0,0}(\Sigma)\},$$

$$I(y,u) = \frac{1}{2}\int_0^T (|Py|^2_{L^2(\Omega)} + |R_A^{1/2}\gamma_n u|^2_{V^0(\Gamma)} + |\gamma_\tau u|^2_{V^0(\Gamma)})\, dt,$$

$$\begin{aligned} Py' &= APy + BMu \quad \text{in} \quad (0,\infty), \\ Py(0) &= \zeta, \qquad\qquad (2) \\ (I-P)y &= (I-P)D_A M\gamma_n u, \end{aligned}$$

where

$$Ay = \frac{1}{\mathrm{Re}}P\Delta y - P((w\cdot\nabla)y) - P((y\cdot\nabla)w),$$

$B = (\lambda I - A)PD_A$ and
$R_A = MD_A^\star(I-P)D_AM + I.$

To this problem, we can apply Riccati optimal control theory. We solve the ARE

$$A^\star\Pi + \Pi A - \Pi B_\tau M^2 B_\tau^\star \Pi + \Pi B_n M R_A^{-1} M B_n^\star \Pi + I = 0$$

for $\Pi = \Pi^\star \geq 0$, define the feedback control

$$u = -MB_\tau^\star \Pi Py - R_A^{-1} M B_n^\star \Pi Py$$

and get the stabilized solution from (2).

This theory can be extended to the fully nonlinear equation with the additional term $(y\cdot\nabla)y$, and the stabilization can even be made exponential such that, if the initial perturbation $y(0) = \zeta$ is small enough,

$$\exists\, C,\, \omega > 0: \quad ||y(t)|| \leq C\, e^{-\omega t}.$$

3. NUMERICAL REALIZATION

We will demonstrate the Riccati-based approach for a standard benchmark problem in flow control: the backward facing step. Here the goal is to minimize the vorticity behind the step by applying a Dirichlet boundary control.

We are going to use the Taylor-Hood finite element Galerkin space discretization from which we will get n-dimensional approximations of the state equations and the ARE.

Solving the ARE is a numerical challenge due to the size of the solution matrix $\Pi_h \in I\!R^{n\times n}$. We are going to use a low-rank Cholesky approximation $\Pi \approx Z_h Z_h^T$ with $Z_h \in I\!R^{n\times r}, r \ll n$, and compute Z_h by a variant of Newton's method for AREs. Our algorithms exploit the structure of the coefficient matrices by alternating direction iteration methods such that the complexity of each Newton step is reduced from $\mathcal{O}(n^3)$ to the complexity for solving the stationary Stokes problem (3; 4).

For the solution of the differential equations, we use the finite element based solver NAVIER (2). It comprises coupling with energy and species transport, phase change problems and capillary free boundary conditions, for example. There are versions for 2d and 3d.

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