SHAPE OPTIMIZATION IN 3D CONTACT PROBLEMS WITH COULOMB FRICTION

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1. INTRODUCTION

Since 1980, a considerable attention of applied mathematicians has been devoted to unilateral contact problems with Coulomb friction, cf. [2] and the references therein. Concerning the static case, our comprehension has reached a fairly satisfactory level. In [1], the authors have developed a numerical approach to a class of optimization problems, where one computes optimal shape of a 2D elastic body in contact with a rigid obstacle which obeys the Coulomb friction law. The problem has been formulated as a mathematical program with equilibrium constraints (MPEC) and solved via the so called implicit programming approach (ImP), cf. [5]. The technique from [1] cannot be, however, extended to the 3D case in a straightforward way. The reason is the nonpolyhedral nature of the subdifferential map of the Euclidean norm in \( \mathbb{R}^n \), whenever \( n \geq 2 \). Whereas the stability and sensitivity analysis of variational inequalities/generalized equations over polyhedral constraint sets has been developed quite deeply so far, much less is known about the nonpolyhedral case. This holds in particular for the generalized equation (GE) modeling the investigated 3D contact problem. Further, also the numerical solution of this GE with a fixed shape of the body (which is the state problem in our MPEC) is substantially more demanding. The main aim of this contribution is to extend the ImP technique of [1] to the 3D case, which requires to discretize this MPEC by finite elements, to construct a fast and precise solver for the state problem and, by using tools of sensitivity analysis, to compute a "subgradient" information, needed in the used nonsmooth optimization method.

2. NUMERICAL APPROACH

Our workhorse in sensitivity analysis is the generalized differential calculus of B. Mordukhovich ([4]) which is applied to the solved (discretized) MPEC along the lines of [3]. The main difficulty arises thereby in the treatment of the generalized equation

\[
\begin{align*}
0 & \in A_{\tau \tau}(x)u_\tau + A_{\tau \nu}(x)u_\nu - l_\tau(x) + \tilde{Q}(u_\tau, \lambda), \\
0 & = A_{\nu \tau}(x)u_\tau + A_{\nu \nu}(x)u_\nu - l_\nu(x), \\
0 & \in u_\nu + x + N_{\mathbb{R}^p_+}(\lambda),
\end{align*}
\]

(1)
defining the discretized state problem. In this model we have to do only with nodes laying on the contact boundary which shape is subject to optimization. The state variable \( y = (u_\tau, u_\nu, \lambda) \in \mathbb{R}^{2p} \times \mathbb{R}^p \times \mathbb{R}^p_+ \), where \( p \) is the number of nodes, \( u_\tau \) is the vector of tangent displacements, \( u_\nu \) is the vector of normal displacements and \( \lambda \) is the multiplier associated with the nonpenetrability constraint

\[
u_\nu + x \geq 0.
\]

(2)

In (1), (2) the control \( x \in \mathbb{R}^p \) specifies the shape of the contact boundary. \( A_{\tau \tau}, A_{\tau \nu}, A_{\nu \tau} \) and \( A_{\nu \nu} \) are blocks of the appropriate restriction of the stiffness matrix which depend on \( x \) in a continuously differentiable way. This holds true also for the vectors \( l_\tau, l_\nu \) reflecting the action of external forces. The multifunction \( Q \) in the first line of (1) is given by

\[
\tilde{Q}(u_\tau, \lambda) = \lambda \bullet \partial j(u_\tau), \quad j(u_\tau) = \mathcal{F} \sum_{i=1}^{p} \|u_\tau^i\|,
\]

where \( u_\tau^i \) is the tangential displacement of the \( i \)th node, \( \| \cdot \| \) is the Euclidean norm in \( \mathbb{R}^2, \mathcal{F} > 0 \).
is the friction coefficient and \( \bullet \) denotes the Hadamard product.

The shape optimization problem is defined as follows:

\[
\begin{align*}
\text{minimize} \quad & f(x, y) \\
\text{subject to} \quad & y \text{ solves the GE (1)} \\
& x \in \omega,
\end{align*}
\]

where \( f \) is the objective and \( \omega \) is the set of admissible controls. Since (1) defines a single-valued and locally Lipschitz map \( S \) assigning \( x \) the corresponding state variable \( y \), problem (3) amounts to the nonsmooth program

\[
\begin{align*}
\text{minimize} \quad & \Theta(x) := f(x, S(x)) \\
\text{subject to} \quad & x \in \omega.
\end{align*}
\]

The step from (3) to (4) is the core of ImP. To solve (4) numerically, one needs to be able to compute at each \( x \in \omega \) the corresponding state variable \( y = S(x) \) and one arbitrary vector \( \xi \) from the Clarke subdifferential of \( \Theta \). In our approach this is done by solving the (regular) adjoint generalized equation.

\[
0 \in \nabla_y f(x, y) + (\nabla_y F(x, y))^T v + \hat{D}^* Q(y, -F(x, y))(v)
\]

in variable \( v \), where \( y = S(x) \) and \( F, Q \) denote the single-valued and the multi-valued part in (1), respectively. \( \hat{D}^* Q \) is the regular coderivative which is replaced sometimes by the limiting coderivative \( D^* Q \), cf.[3]. Having computed a solution \( v \) of (5), we use the formula

\[
\xi = \nabla_x f(x, y) + (\nabla_x F(x, y))^T v
\]

to arrive at the desired subgradient. As nonsmooth optimization solver, we use the classical bundle-trust algorithm from (6). We provide numerical results of several test examples to illustrate the properties of the proposed approach.

3. CONCLUSION

The investigated optimization problem belongs to the hardest MPECs ever solved. This concerns both the applied tools from variational analysis as well as numerical complexity and dimensionality. In the numerical treatment some other alternatives are available and deserve a proper testing.