A Collocation Method for Quadratic Control Problems Governed by Ordinary Elliptic Differential Equations

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1. INTRODUCTION

We consider the one-dimensional elliptic optimal control problem

$$\text{(CP)} \quad \min \frac{1}{2} \int_0^T |z(t) - z_d(t)|^2 + \nu\, |u(t)|^2\, dt$$

s.t.

$-\ddot{z}(t) + Az(t) = Bu(t) + e(t)$ for a.a. $t \in [0, T]$,

$z(0) = z(T) = 0$,

$a \le u(t) \le b$ for a.a. $t \in [0, T]$,

where $u \in L_2(0, T; \mathbb{R}^m)$, $z, z_d \in W_2^2(0, T; \mathbb{R}^n)$, $\dot{e} \in BV(0, T; \mathbb{R}^n)$, $A \in \mathbb{R}^{n \times n}$ is symmetric and positive semidefinite, $B \in \mathbb{R}^{n \times m}$ and $a, b \in \mathbb{R}^m$, $a < b$.

2. DISCRETIZATION OF THE STATE EQUATION

For the discretization of the state equation

$$\begin{aligned} -\ddot{z}(t) + Az(t) &= y(t) \text{ for a.a. } t \in [0, T], \\ z(0) = z(T) &= 0, \end{aligned} \tag{1}$$

we use a uniform grid

$$G = \{t_i = ih \mid i = 0, \dots, N\} \tag{2}$$

with mesh size $h = T/N$, $N \ge 2$. By $S_h = S_h(y)$ we denote the quadratic spline with knots t_i, $i = 0, \dots, N$, defined by the collocation and boundary conditions

$$\begin{aligned} -\ddot{S}_h(t_i) + AS_h(t_i) &= y(t_i),\ i = 0, \dots, N, \\ S_h(0) = S_h(T) &= 0. \end{aligned}$$

If z ist the solution of the state equation (1) and $\dot{y} \in BV(0, T; \mathbb{R}^n)$, then

$$\|z - S_h\|_\infty \le c\, h^2$$

with a constant c independent of h (see Sendov (5), Theorem 7.3).

3. DISCRETIZATION OF THE CONTROL PROBLEM

We define $U_{\text{ad}} = \{u \in L_2(0, T; \mathbb{R}^m) \mid a \le u(t) \le b \text{ for a.a. } t \in [0, T]\}$. For a function f continuous on $[0, T]$ we define

$$\|f\|_h = \sqrt{h \sum_{i=0}^{N} |f(t_i)|^2}\,.$$

Let $V_h(0, T; \mathbb{R}^m)$ be the space of continuous, piecewise linear functions on the grid (2). Using the operator $\mathcal{S}_h$ we discretize problem (CP) in the following way:

$$\text{(CP)}_h \quad \min \frac{1}{2} \|\mathcal{S}_h(Bu_h + e) - z_d\|_h^2 + \nu\, \|u_h\|_h^2$$

s.t. $u_h \in U_h^{ad} = U_{\text{ad}} \cap V_h(0, T; \mathbb{R}^m)$.

Problem (CP)_h has a unique solution $\bar{u}_h$.

4. ERROR ESTIMATES

First we derive a result on discrete quadratic convergence for the solutions $\bar{u}_h \in V_h(0, T; \mathbb{R}^m)$ of the problems (CP)_h.

Theorem 1. *Let $\bar{u}$ be the solution of* (CP2) *with $\dot{\bar{u}} \in BV(0, T; \mathbb{R}^m)$ and $\bar{u}_h \in V_h(0, T; \mathbb{R}^m)$ the solution of the discrete problem* (CP)_h. *Then*

$$\|\bar{u} - \bar{u}_h\|_h \le c\, h^2, \tag{3}$$

holds true with a constant c independent of h.

The *continuous* error $\|\bar{u} - \bar{u}_h\|_\infty$ is only of order $3/2$. Therefore, we adopt the idea of Meyer/Rösch (4) (see also (1), (2)) to construct a new feasible control by

$$\tilde{u}_h = \Pi_{[a,b]} \left(-\frac{1}{\nu} B^\mathsf{T} p_h(\bar{u}_h) \right), \tag{4}$$

for which we can prove continuous convergence of order 2.

Theorem 2. *Let $\bar{u}$ be the solution of problem* (CP2) *with $\dot{\bar{u}} \in BV(0,T;\mathbb{R}^m)$ and $\bar{u}_h \in V_h(0,T;\mathbb{R}^m)$ the solution of the discrete problem* (CP)$_h$. *Then for the control $\tilde{u}_h$ defined by* (4) *we have the continuous error estimate*

$$\|\bar{u} - \tilde{u}_h\|_\infty \leq c\, h^2$$

with a constant c independent of h.

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