Cross Gramians for Nonlinear Systems

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1. INTRODUCTION

We aim at extending the approximate balancing method presented in (Sorensen et al., 2002; Aldaheri, 1991). It deals with model reduction for linear symmetric systems in an efficient way, based on solving a Sylvester equation whose solution is the so-called cross-Gramian (e.g. see (Fernando et al., 1983)). The eigenvalues of the cross Gramian are the Hankel singular values of the system. The advantages of this method, in comparison to the usual balancing procedure, are that it requires solving only one Sylvester equation and that it avoids the balancing procedure, being more efficient from computational point of view.

We study the notion of cross Gramians for nonlinear gradient systems, which are the extension of the notion of symmetric systems. We use the prolongation and gradient extension associated to the gradient system, as in (Cortes et al., 2005). The cross Gramian is given for the variational system (part of the prolongation) associated to the original nonlinear gradient system. We obtain linearization results that precisely correspond to the notion of a cross Gramian for symmetric linear systems. Furthermore, starting from the work in (Fujimoto et al., 2005), first steps towards relations with the singular value functions of the nonlinear Hankel operator are studied and yield promising results.

2. LINEAR SYSTEMS CASE

Definition 1 (Sorensen et al., 2002) The cross Gramian X of a linear system $\dot{x} = Ax + Bu,\ y = Cx$, is defined as the solution of the Sylvester equation:

$$AX + XA + BC = 0. \tag{1}$$

If the system is asymptotically stable, then :

$$X = \int_0^\infty e^{At} BC e^{At} dt.$$

□

Definition 2 (Aldaheri, 1991) A linear system is symmetric if and only if $H(s) = H^T(s)$, where $H(s) = C(sI - A)^{-1}B$, or equivalently, there exists T, invertible and symmetric s.t.

$$TA = A^T T,\ TB = C^T.$$

□

The cross Gramian has some interesting properties:

Theorem 3 (Sorensen et al., 2002) If the linear system is symmetric, asymptotically stable and minimal (controllable and observable), then:

$$X = T^{-1}M = WT \text{ and } X^2 = WM,$$

where $M > 0,\ W > 0$ are the observability and controllability Gramians, respectively. □

3. GRADIENT SYSTEMS AND LINEARIZATION

Definition 4 (Cortes et al., 2005) A nonlinear affine system

$$\begin{cases} \dot{x} = f(x) + g(x)u \\ y = h(x) \end{cases},\ x \in M,\ u, y \in \mathbb{R}^p \tag{2}$$

is called a gradient system if,

1. there exists a pseudo-Riemannian metric on the manifold M, with the associated matrix $G(x)$, symmetric and invertible for all $x \in M$;
2. there exists a smooth potential function $V : M \to \mathbb{R}$,

such that (2) can be written as:

$$\begin{cases} \dot{x} = -G^{-1}(x)\dfrac{\partial^T V}{\partial x}(x) + G^{-1}(x)\dfrac{\partial^T h}{\partial x}(x)u \\ y = h(x) \end{cases} \tag{3}$$

□

For the nonlinear system two associated systems are defined:

- the prolongation :

$$\begin{cases} \dot{x} = f(x) + g(x)u \\ \dot{v} = \dfrac{\partial f(x)}{\partial x} v + \displaystyle\sum_{j=1}^{m} u_j \dfrac{\partial g_j(x)}{\partial x} v + g(x)u_p \\ y = h(x),\ y_p = \dfrac{\partial h(x)}{\partial x} v \end{cases}, \tag{4}$$

- the gradient extension:

$$\begin{cases} \dot{x} = f(x) + g(x)u \\ \dot{p} = \dfrac{\partial^T (f(x) + g(x)u)}{\partial x} p \\ +\mathcal{F}(g_{ij}(x), \frac{\partial g_{ij}(x)}{\partial x_k}, f_k(x), p) + \dfrac{\partial h(x)}{\partial x} u_g, \\ y = h(x),\ y_g = g^T(x)p,\ i,j,k = 1...n \end{cases} \tag{5}$$

Theorem 5 (Cortes et al., 2005) Let (2) be locally observable. Assume that (5) exists and is well defined. Then, under additional technical conditions, (2) is a gradient control system, as in (3), if and only if the prolonged system Σ_p and the gradient extension Σ_g have the same input-output behaviour. □

Lemma 6 (Cortes et al., 2005, Lemma 5.5, 5.6) If (2) is a gradient control system, then there exists a diffeomorphism $\phi(x, v) = (x, G(x)v)$, such that $(x, p) = \phi(x, G(x)v)$. □

Linearizing (3), around an equilibrium x_0, a linear gradient (symmetric) system is obtained, whose metric is $T = G(x_0)$. Assume that the observability function and the controllability function of (2), $L_o(x)$ and $L_c(x)$, respectively, exist and are positive definite. Also, suppose that the observability Gramian M and the controllability Gramian W of the linearized system exist and are positive definite. Then $(\partial^2 L_o(x)/\partial x^2) = M$ and $(\partial^2 L_c(x)/\partial x^2) = W^{-1}$. The symmetry of the linearized system, implies, according to Theorem 3, that, near x_0:

$$G^{-1}(x) \frac{\partial^2 L_o}{\partial x^2}(x) = \left(\frac{\partial^2 L_c}{\partial x^2}(x) \right)^{-1} G(x)$$

Moreover, the linearizations of (4) and around an equilibrium $(x_0, 0)$, yield two linear systems dual to each other, in the variables v, p. If (2) is gradient, then $p = Tv$, $T = G(x_0)$. Thus, a study of the variational system (v part of (4)) is motivated.

4. NONLINEAR CROSS GRAMIAN

Denote by Σ'_p, the variational part of (4) and Σ'_g, the p part of (5), of (2) which is assumed gradient. According to Lemma 6, $p = G(x)v$. Assume that $L_o(x, v) = \frac{1}{2} v^T M(x) v$ exists and is positive definite, where the entries of $M(x)$ are smooth functions. Then, $M(x)$ satisfies

$$\begin{aligned} & p^T G^{-1}(x) M(x) \frac{\partial f(x)}{\partial x} v + \frac{1}{2} p^T g(x) g^T(x) p \\ & = \frac{\partial L_o(x, v)}{\partial x} f(x) - v^T \frac{\partial^2 L_o(x, v)}{\partial v \partial x} f(x) \end{aligned} \tag{6}$$

We call $\mathcal{X}(x) = G^{-1}(x)M(x)$ the cross Gramian of Σ'_p.

Remark 7 In the linear case, (6) becomes:

$$T^{-1} M \cdot A + \frac{1}{2} BC = 0,$$

where $X = T^{-1}M$ is the cross Gramian. □

Conjecture 8 For a nonlinear gradient system with the associated variational system Σ'_p, if λ_i, $i = 1, ..., n$ satisfy

$$\frac{\partial L_o}{\partial x}(x(0)) = \lambda \frac{\partial L_c}{\partial x}(x(0)),$$

then they are the squared eigenvalues of $\mathcal{X}(x)$.

Since, the λ's are related to (4), associated to (2), it means that if they are related to the eigenvalues of the cross Gramian, the Hankel singular values can be obtained from solving an eigenvalue problem for $\mathcal{X}(x)$. Then, the metric and the observability function provide the singular values of the system, similar to the linear case.

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