Two-person zero-sum differential games

Michel C. Delfour

Centre de recherches mathématiques
Université de Montréal
CP 6128, succ. Centre-ville
Montréal (Qc), Canada H3C 3J7.
e-mail: delfour@crm.umontreal.ca

Keywords: Linear quadratic differential game, saddle point, value of a game, Riccati differential equation, open loop and closed loop strategies, conjugate point, blow-up time

In *differential zero-sum two-player games* the first player tries to minimize and the second player to maximize a utility function that depends on a state variable whose dynamics is governed by a system of differential equations. Two classical approaches are via *open loop* and *closed loop strategies* for the players. In this paper, we shall mainly restrict ourselves to the open loop case and a perfect knowledge of the *state*.

(M. C. Delfour and S. K. Mitter 1969) studied the dynamical Min-Sup problem for the following perturbed control process in $\mathbb{R}^n$

$$\frac{dx}{dt}(t) = A(t)\,x(t) + f(t, u(t)) + g(t, v(t)), \quad (1)$$

where $A(t)$ is a $n \times n$ measurable and bounded matrix on $[0, T]$, f is in C^1 in $\mathbb{R}^{1+m}$ and g is in C^1 in $\mathbb{R}^{1+k}$ (n, m, and k are integers ≥ 1), and furthermore, (i) the initial state x_0 at time 0 is given, (ii) the admissible controllers $\mathcal{F}$ consist of all Lebesgue measurable functions $t \mapsto u(t)$ on the compact interval $[0, T]$ such that $u(t) \in U$, (almost everywhere on $[0, T]$), where U is a compact set in $\mathbb{R}^m$, (iii) the admissible disturbances $\mathcal{G}$ consist of all Lebesgue measurable functions $t \mapsto v(t)$ on the compact interval $[0, T]$ such that $v(t) \in V$, almost everywhere on $[0, T]$, where V is a compact set in $\mathbb{R}^k$, (iv) the cost function for each admissible u and v is given by $C(u, v) = G(x(T))$, where G is a continuous function in $\mathbb{R}^n$.

The fundamental theory of closed loop two-player zero-sum LQ games was given in (P. Bernhard 1979) followed by the seminal book in 1991 of (T. Başar and P. Bernhard 1995) that covered the H^∞-theory. They considered two-player zero-sum game over the finite time interval $[0, T]$ characterized by the quadratic *utility function*

$$C_{x_0}(u, v) \stackrel{\text{def}}{=} Fx(T) \cdot x(T) + \int_0^T Q(t)x(t) \cdot x(t) + |u(t)|^2 - |v(t)|^2 \, dt, \quad (2)$$

where x is the solution of the linear differential system, the so-called *state equation*

$$\frac{dx}{dt}(t) = A(t)x(t) + B_1(t)u(t) + B_2(t)v(t) \quad \text{a.e. in } [0, T], \qquad x(0) = x_0, \quad (3)$$

x_0 is the *initial state* at time $t = 0$, $u \in L^2(0, T; \mathbb{R}^m)$, $m \geq 1$, is the strategy of the first player, and $v \in L^2(0, T; \mathbb{R}^k)$, $k \geq 1$, is the strategy of the second player. F is an $n \times n$-matrix and A, B_1, B_2, and Q are matrix-functions of appropriate order that are measurable and bounded almost everywhere in $[0, T]$. Moreover, $Q(t)$ and F are positive semi-definite.

Given an initial state x_0 in $\mathbb{R}^n$ at time $t = 0$, the game is said to achieve its *open loop lower value* if

$$v^-(x_0) \stackrel{\text{def}}{=} \sup_{v \in L^2(0.T; \mathbb{R}^k)} \; \inf_{u \in L^2(0.T; \mathbb{R}^m)} C_{x_0}(u, v)$$

is finite. It is said to achieve its *upper value* if

$$v^+(x_0) \stackrel{\text{def}}{=} \inf_{u \in L^2(0.T; \mathbb{R}^m)} \; \sup_{v \in L^2(0.T; \mathbb{R}^k)} C_{x_0}(u, v)$$

is finite. By definition $v^-(x_0) \leq v^+(x_0)$. The game is said to achieve its *open loop value* if its open loop lower value $v^-(x_0)$ and upper value $v^+(x_0)$ are finite and $v^-(x_0) = v^+(x_0)$. The *open loop value* of the game will be denoted by $v(x_0)$. A pair $(\bar{u}, \bar{v})$ in $L^2(0, T; \mathbb{R}^m) \times$

$L^2(0,T;\mathbb{R}^k)$ is an *open loop saddle point* of $C_{x_0}(u,v)$ in $L^2(0,T;\mathbb{R}^m) \times L^2(0,T;\mathbb{R}^k)$ if for all u in $L^2(0,T;\mathbb{R}^m)$ and all v in $L^2(0,T;\mathbb{R}^k)$ $C_{x_0}(\bar{u},v) \leq C_{x_0}(\bar{u},\bar{v}) \leq C_{x_0}(u,\bar{v})$. An open loop saddle point coincides with the classical notion of a *Nash equilibrium*.

The very nice work of (P. Zhang 2005-1) established the equivalence between the finiteness of the open loop value of a two-player zero-sum LQ game and the finiteness of its open loop lower and upper values *without a priori positive semidefiniteness assumption* on the matrices entering in the utility function, that is $Q(t)$ and F are not necessarily positive semi-definite. It means that the *duality gap*, that is the difference between the upper and the lower values of the game, is either 0 or $+\infty$. The reader is referred to the above references for a detailed bibliography of the vast and rich literature on dynamical games.

In a recent paper (M. C. Delfour 2007) completed and sharpened the results of (P. Zhang 2005-1) for the finiteness of the lower value of the game by providing a set of necessary and sufficient conditions that emphasizes the *feasibility condition*: $(0,0)$ is a solution of the open loop lower value of the game for the zero initial state.

Then he shows that, under the assumption of an open loop saddle point in the time horizon $[0,T]$ for all initial states, there is an open loop saddle point in the time horizon $[s,T]$ for all initial times s, $0 \leq s < T$, and all initial states at time s. From this he gets an *optimality principle*, adapts the *invariant embedding approach* to construct the decoupling symmetrical matrix function $P(s)$, and shows that it is an $H^1(0,T)$ solution of the matrix Riccati differential equation. Thence an open loop saddle point in $[0,T]$ yields closed loop optimal strategies for both players.

Furthermore, a necessary and sufficient set of conditions for the existence of an open loop saddle point in $[0,T]$ for all initial states is the convexity-concavity of the utility function and the existence of an $H^1(0,T)$ symmetrical solution to the matrix Riccati differential equation. As an illustration of the cases where the open loop lower/upper value of the game is $-\infty/+\infty$, two informative examples of solutions to the Riccati differential equation with and without blow-up time are worked out.

In this paper we first go back to the case where the lower value of the game is finite and the upper value is infinite. In general, it is difficult to make sense of a Riccati differential equation. Yet, (P. Bernhard 1979) studied a family of games where it is still possible to get such an equation provided escape times are allowed at a finite number of times. For that family there is a closed loop-closed loop saddle point. Some thoughts and hopefully new insights will be presented.

Finally, we shall extend the above results to infinite dimensional parabolic systems. For related work the reader is referred to (I. Lasiecka and R. Triggiani 2000) and their bibliography.

REFERENCES

[T. Başar and P. Bernhard 1995] T. Başar and P. Bernhard, *H^∞-Optimal Control and Related Minimax Design Problems: A Dynamical Game Approach*, 2nd ed., Birkhäuser, Boston, MA, 1995.

[P. Bernhard 1979] P. Bernhard, *Linear-quadratic, two-person, zero-sum differential games: necessary and sufficient conditions*, JOTA 27 (1979), 51–69.

[M. C. Delfour 2007] M. C. Delfour *Linear-Quadratic Differential Games: Saddle Point and Riccati Differential Equation*, SIAM J. on Control and Optim., to appear.

[M. C. Delfour and S. K. Mitter 1969] M. C. Delfour and S. K. Mitter, *Reachability of perturbed linear systems and* $\min\sup$ *problems*, SIAM J. Control Optim. **7** (1969), 521–533.

[I. Lasiecka and R. Triggiani 2000] I. Lasiecka and R. Triggiani, *Control Theory for Partial Differential Equations: continuous and approximation theories*, Vol I, Cambridge University Press, Cambridge, U.K. 2000

[P. Zhang 2005-1] P. Zhang, *Some results on two-person zero-sum linear quadratic differential games*, SIAM J on Control and Optim. 43, no. 6 (2005), 2157–2165.

[P. Zhang 2005-2] P. Zhang, G. Zheng, Y. Xu, and J. Xi, *The Riccati equation of game type*, in “Proceedings of the 24th Chinese Control Conference”, Guangzhou, China, July, 2005, pp. 573–577.