# Problems of mathematical finance by stochastic control methods

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## 1. Introduction

The purpose of the talk is to present main ideas of mathematics of finance using the stochastic control methods. There is an interplay between stochastic control and mathematics of finance. From one hand side stochastic control is a powerful tool to study financial problems. On the other hand financial applications have stimulated development in several research subareas of stochastic control in the last two decades.

## 2. Pricing of financial derivatives

One of the classical problems of mathematics of finance is pricing of financial derivatives. At a given time $T$ called maturity the buyer of a financial instrument collects a gain which a random variable $H$ (called a contingent claim). We would like to evaluate the price of $H$ at the initial time. One can look at this price from the perspective of the seller or the buyer. An acceptable price for the seller is a such price that for the amount he obtains at time 0 he is able (providing he invests it in an optimal way) to get at least the compensation for $H$, which he is supposed to deliver to the buyer. This investment to hedge $H$ forms a stochastic optimal control problem. We would like to find the smallest initial capital $v$, which invested in an optimal way gives us at least the value $H$ at time $T$. Denote by $p_s(H)$ the minimal seller price. The buyer price is the value $v$ such that if he starts with initial capital $-v$ and invests it in an optimal way, then at time $T$ and the value of his portfolio plus his gain $H$ is nonnegative. Such maximal $v$ forms so called the buyer price $p_b(H)$ and is the maximal price acceptable for the buyer. Clearly $p_b(H) \leq p_s(H)$. The interval $[p_b(H), p_s(H)]$ is called an absence of arbitrage interval and any price from this interval is acceptable in the sense that neither seller nor buyer is able to obtain a positive gain without risk at time $T$ (which we call an arbitrage). In a particular situations when $p_b(H) = p_s(H)$ for all bounded $H$ we say that the market is complete which in turn corresponds to the fair price or fair game between the seller and the buyer. If we assume that the market does not allow an arbitrage (which is a standard assumption) then the buyer and seller prices have an interpretation as the minimum or maximum over the set of martingale measures $\mathcal{Q}$ of the expected value of $H$ with respect to martingale measures. The martingale measure is an equivalent measure to the original probability measure under which the asset prices of our market become martingales i.e integrable stochastic processes such that the conditional expectation of their value at time $t+1$ given an information till time $t$ is equal to their value at time $t$. It should be pointed out that such nice representation of the seller and buyer prices is valid only when we assume that there are no transaction costs on the market. The case of proportional (to the value of transaction) transaction costs has been studied recently by a number of well known mathematicians and still seems to be open in various aspects. In practice we usually pay fixed plus proportional transaction costs which are much harder to analyze. Furthermore frequently we can expect, in the case of larger transaction, to pay smaller proportional transaction costs which makes the transaction costs to be concave. The recent studies on liquidity effects lead even to convex transaction costs. In the last three cases we have only partial results concerning particular models. The general approach seems to be open.

## 3. Credit risk

The model and considerations in the previous section were based on the assumption that if the transaction is made at time $0$ the contingent claim $H$ is delivered at time $T$. It may happen however that this delivery is subject to a certain risk called default. When default happens before or at time $T$ then instead of $H$ only its portion called recovery claim is delivered at time $T$. From mathematical point of view we may consider two cases: the case when default time is a predictable stopping time basing on the available information and the case when it is an unpredictable stopping time with a known intensity. In both cases we have similar questions for defaultable contingent claims: what is the price for such a claim if there are no or there are transaction costs. Consequently the pricing problems formulated in the section 2 can be considered.

## 4. Term structure models

In the models above we consider investments in assets or in banking account. An extension of the market is to consider investments in bonds. We denote by $B(t,T)$ the price at time $t$ of the zero coupon bond paying one unit at time $T$. In so called single factor approach the price $B(t,T)$ is equal to the conditional expected value of the exponent of the negative value of the integral from $t$ to $T$ with respect to short term interest rate $r(t)$, which may be a solution to a certain stochastic differential equation. In more recent models following Heath Jarrow Morton methodology $B(t,T)$ is an exponent of the integral from $t$ to $T$ of a two factor function $f(t,s)$ called an instantaneous forward rate, which for fixed $s$ as a function of $t$ is a solution to a stochastic differential equation. One can again consider the problems of pricing of bond market derivatives or any contingent claims using investments on such extended market.

## 5. Portfolio selection

A vast literature in mathematics of finance is devoted to portfolio selection problems. We are looking for a portfolio maximizing of certain utility function. Any function which is concave and increasing and is a function of our consumption or value of our wealth process may be used to measure our satisfaction (our utility). The problem of utility maximization can be considered both for dynamical models, where growth of asset prices depends on time and we are looking for an investment strategy maximizing utility over certain time horizon, and for static models, where basing on historical data we model one step growth rate and maximize portfolio over one time step. In the case of dynamic models we maximize the utility of our consumption together with the utility of the terminal wealth process. Another problem is to maximize the growth of portfolio which is a logarithmic utility function of the wealth process. Portfolio selection models can be also used to price financial derivatives using e.g. indifference price, the price which guarantee the same expected value of the terminal value of the utility function of the wealth process, with initial capital diminished by the price, plus the contingent claim as in the case when we start with non diminished initial capital and at the terminal time we do not obtain the contingent claim.

## 6. Risk

One of the major problems of modern applied mathematics is risk modelling. First of all it is not clear what is risk and how to measure it. Historically the first approach to risk was introduced in 1952 by Harry Markowitz, a later Nobel Price winner in economy in 1990. He measured risk as a variance the random portfolio wealth rate of return. The problem was to maximize over one time step (static portfolio selection) the expected portfolio rate of return with variance considered as a measure of risk not exceeding a certain level. In this problem we practically have two cost functionals one measuring the expected portfolio rate of return and another one corresponding to risk measured in a form of the portfolio variance. The same problem can be considered in dynamic setting. Instead of two cost functionals one also can consider so called risk sensitive cost functional, which measures the expected value plus variance with a certain weight called risk factor. An alternative is to maximize the portfolio wealth rate of return with a restriction on the risk introduced by the Value at Risk (VaR), which is a certain quantile restriction on the lower bound of portfolio or other measures of risk e.g. conditional VaR.