VARIATIONAL ANALYSIS IN OPTIMIZATION AND CONTROL

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Variational analysis has been recognized as a rapidly growing and fruitful area in mathematics concerning mainly the study of optimization and equilibrium problems, while also applying perturbation ideas and *variational principles* to a broad class of problems and situations that may be not of a variational nature. It can be viewed as a modern outgrowth of the classical calculus of variations, optimal control theory, and mathematical programming with the focus on *perturbation/approximation* techniques, sensitivity issues, and applications; see (1; 2; 3)

One of the most characteristic features of modern variational analysis is the intrinsic presence of *nonsmoothness*, i.e., the necessity to deal with nondifferentiable functions, sets with nonsmooth boundaries, and set-valued mappings. Nonsmoothness naturally enters not only through initial data of optimization-related problems (particularly those with inequality and geometric constraints) but largely via *variational principles* and other optimization, approximation, and perturbation techniques applied to problems with even smooth data. In fact, many fundamental objects frequently appearing in the framework of variational analysis (e.g., the distance function, value functions in optimization and control problems, maximum and minimum functions, solution maps to perturbed constraint and variational systems, etc.) are inevitably of nonsmooth and/or set-valued structures requiring the development of new forms of analysis that involve *generalized differentiation*.

It is important to emphasize that even the simplest and historically earliest problems of *optimal control* are *intrinsically nonsmooth*, in contrast to the classical calculus of variations. This is mainly due to pointwise constraints on control functions that often take only discrete values as in typical problems of automatic control, a primary motivation for developing optimal control theory. Optimal control has always been a major source of inspiration as well as a fruitful territory for applications of advanced methods of variational analysis and generalized differentiation.

In this talk we discuss some new trends and developments in variational analysis and its applications mostly based on the author's recent 2-volume book (1; 2). Generalized differentiation lies at the heart of variational analysis and its applications. We systematically develop a *geometric dual-space approach* to generalized differentiation theory revolving around the *extremal principle*, which can be viewed as a local *variational* counterpart of the classical convex separation in nonconvex settings. This principle allows us to deal with *nonconvex* derivative-like constructions for sets (normal cones), set-valued mappings (coderivatives), and extended-real-valued functions (subdifferentials). These constructions are defined directly in dual spaces and, being nonconvex-valued, cannot be generated by any derivative-like constructions in primal spaces (like tangent cones and directional derivatives). Nevertheless, our basic nonconvex constructions enjoy *comprehensive/full calculus*, which happens to be significantly better than those available for their primal and/or convex-valued counterparts. The developed generalized differential calculus based on variational principles provides the *key tools* for various applications.

Observe to this end that dual objects (multipliers, adjoint arcs, shadow prices, etc.) have always been at the center of variational theory and applications used, in particular, for formulating the main optimality conditions in the calculus of variations, mathematical programming, optimal control, and economic modeling. The usage of variations of optimal solutions in primal spaces can be considered just as a convenient tool for deriving necessary optimality conditions. There are no essential restrictions in such a "primal" approach in smooth and convex frameworks, since primal and dual derivative-like constructions are equivalent for these classical settings. It is not the case any more in the framework of modern variational analysis, where even *nonconvex primal space* local approximations (e.g., tangent cones) inevitably yield, *under duality, convex sets* of normals and subgradients. This convexity of dual objects leads to significant restrictions for the theory and applications. Moreover, there are many situations particularly identified in this book, where primal space approximations simply cannot be used for variational analysis, while the employment of dual space constructions provides comprehensive results.

In this talk we pay the main attention to discussions of the basic constructions of generalized differentiation in variational analysis and their applications to problems of *nonsmooth constrained optimization* and *optimal control*. We present complete characterizations of *Lipschitzian stability* and *metric regularity* of constraint and variational systems and their applications to *sensitivity analysis* with respect to perturbations. Then we discuss necessary optimality conditions for some remarkable classes of optimization problems including *nondifferentiable programming* with functional and geometric constraints and rather new while well-recognized classes of mathematical programs and multiobjective optimization problems with the so-called *equilibrium constraints*, which closely relate to problems of *bilevel programming* particularly considered in the talk. Finally, we consider optimal control systems governed by *evolution/differential inclusions* and present new necessary optimality conditions for them in generalized *Euler-Lagrange* and *Hamiltonian* forms. Our approach to optimal control of systems with continuous-time dynamics is based of *discrete approximations*, which provides efficient tools of analysis from both numerical and qualitative viewpoints. It time permits, we discuss particular applications of the results obtained to optimal control systems with continuous-time dynamics described by *ordinary differential*, *functional differential*, and *partial differential equations*.

REFERENCES

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