THEORY AND APPLICATIONS OF OPTIMAL BANG-BANG AND SINGULAR CONTROL PROBLEMS

Helmut Maurer

Universität Münster, Institut für Numerische und Angewandte Mathematik,
Einsteinstr. 62, 48149 Münster, Germany,
Email: maurer@math.uni-muenster.de

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1. OPTIMAL CONTROL PROBLEMS WITH CONTROL APPEARING LINEARLY

We study optimal control problems of the following form: determine a piecewise continuous (measurable) control $u : [0, t_f] \to I\!R^m$ and a state trajectory $x : [0, t_f] \to I\!R^n$ that minimize the cost functional of Mayer type,

$$J(x, u, t_f) := g(x(t_f), t_f),$$

subject to the dynamics, boundary conditions and control-state constraints

$$\begin{aligned} &\dot{x}(t) = f(x(t), u(t), t), \quad 0 \le t \le t_f\,, \\ &\varphi(x(0), x(t_f)) = 0, \\ &C(x(t), u(t)) \le 0, \quad 0 \le t \le t_f\,. \end{aligned}$$

The augmented Hamiltonian is given by

$$H(x, u, \lambda, t) = \lambda f(x, u, t) + \mu C(x, u),$$

where $\lambda \in I\!R^n$ denotes the adjoint variable and μ is the multiplier for the control-state constraint. For this control problem, second–order sufficient conditions, sensitivity analysis and real–time control techniques have been extensively studied in the literature under the assumption that the *strict Legendre condition* $H_{uu}[t] \ge cI_m\,,\ c > 0$, holds; c.f., e.g., Dontchev, Hager [3], Malanowski, Maurer [7], Büskens, Maurer [2], Maurer, Augustin [9].

The situation is different for optimal control problems where all control components appear linearly. In this case, the strict Legendre conditions is violated. The dynamics then has the form

$$\dot{x}(t) = f_1(x(t), t) + f_2(x(t))u(t),$$

where $f_1(x, t)$ is a n–vector and $f_2(x, t)$ is a $n \times m$–matrix, and the control constraints are assumed to be simple box constraints

$$u_{i,\min} \le u_i(t) \le u_{i,\max}\,, \quad i = 1, ..., m.$$

The *switching function* is defined by

$$\begin{aligned} &\sigma(x, \lambda, t) = \lambda f_2(x, t), \\ &\sigma[t] = \sigma(x(t)), \lambda(t)), t) = (\sigma_1[t], ..., \sigma_m[t]). \end{aligned}$$

Then the optimal control which minimizes the Hamiltonian is characterized by

$$u_i(t) = \left\{ \begin{array}{lll} u_{i,\min}, & \text{if} & \sigma_i[t] > 0 \\ u_{i,\max}, & \text{if} & \sigma_i[t] < 0 \\ \text{singular}, & \text{if} & \sigma_i[t] = 0 \end{array} \right\}$$

for $i = 1, ..., m$. If the switching function $\sigma_i[t]$ has only isolated zeros in $[0, t_f]$, then $u_i(t)$ is called a *bang–bang* control component.

2. BANG–BANG CONTROL

Assume that every component $u_i(t)$ of the optimal control is bang–bang and that there are only finitely many switching times which are ordered as $0 < t_1 < ... < t_k < ... < t_s < t_f$. Such a bang-bang control can be computed by solving an *induced optimization problem*, where the switching times t_k, $(k = 1, ..., s)$ are taken as optimization variables. It has been shown in Agrachev, Stefani, Zezza [1] and Osmolovskii, Maurer [11–13] that second order sufficient conditions (SSC) hold for the *bang-bang control problem* provided that SSC hold for the induced optimization problem and, moreover, the switching function satisfies the so–called strict bang–bang property. A related type of sufficient condition has been derived in Ledzewicz, Schättler [6].

An interesting byproduct of the optimization approach is the fact that the well–known sensitivity results for finite–dimensional optimization problems apply to bang–bang control problems, since the strict bang–bang property is stable with respect to perturbations. Numerical time–scaling techniques for verifying SSC and computing parametric sensitivity derivatives have been developed in Maurer et al. [9]. In this talk, we present two practical examples illustrating the numerical techniques and the sufficiency test: time–optimal control of a van der Pol oscillator [11] and control of a semiconductor laser [4].

3. SINGULAR CONTROL

For singular control problems, sufficient otpimality conditions have been obtained only in special cases, e.g., for *totally* singular controls. Here, we concentrate on the case where the singular control can be obtained in *feedback form* $u = u_{\rm sing}(x,t)$. This property holds in many practical examples. To compute a control that is a combination of bang–bang and singular arcs, we solve an induced optimization problem, where switching times of bang–bang arcs and junction times with singular arcs are optimized simultaneously. This numerical approach is illustrated on three examples: (a) van der Pol oscillator [14] (b) Goddard problem [8,14], (c) fedbatch fermentation problem [5,14].

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