SHAPE TUBE METRIC, GEODESIC EQUATION

Jean-Paul Zolésio

CNRS and INRIA, France

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1. MOVING DOMAIN

The Courant metric in shape analysis (16) is extended here to classes of non smooth subsets in D. The intrinsic tube analysis which is evoked here is developped in (25), (24). The characteristic function of Q is $\zeta \in L^\infty(I \times D)$ verifying $\zeta = \zeta^2$ and $\zeta(t) = \chi_{\Omega_t}$ where the measurable set Ω_t is defined in D up to a zero measure subset. That theory can be extended to boundaries with the approach of (2). In the second part we adopt the eulerian modeling (5; 16; 8; 11) which has been extended to non smooth vector fields in (17; 25; 24; 8)... Making use of the transverse field approach (8; 4; 18) we derive the euler equation for the geodesic-tube which has been presented in several image anlysis conferences ("Shape Space" IMA , march 06, MIA06 Paris, Obergurgl ...) with application developed with L. Blanchard (26). The technic is inspired from (7). Following (17), (23), we consider tubes which are continuous with respect to the the $L^1(D)$ topology and with time integrable perimeter, then we introduce the set of characteristic functions

$$\mathcal{P}_D = \ \{ \ \Omega \subset D, \ \chi_\Omega \in BV(D) \ \}$$

and with

$$\mathcal{H}^c = C^0([0,1], L^1(D, \{0,1\})) \cap L^1(\,0,1, BV(D)\,)$$

Associated with any subset $\Omega_0 \in \mathcal{P}_D$, the family

$$\mathcal{O}_{\Omega_0} = \{ \ \Omega \in \ \mathcal{P}_D \ s.t.$$
$$\exists \zeta \in \mathcal{H}^c, \ s.t., \quad \zeta(0) = \chi_{\Omega_0}, \ \zeta(1) = \chi_\Omega \quad \}$$

Associated with any two sets $\Omega_i \in \mathcal{O}_{\Omega_0}$, the non empty set of connecting tubes is :

$$T(\Omega_1, \Omega_2) = \{\zeta \in \mathcal{H}^c, \ \zeta(0) = \chi_{\Omega_1},$$
$$\zeta(1) = \chi_{\Omega_2} \ \}$$

The set of jump lines of $\zeta(t) \in BV(D)$ is denoted by Γ_t. we consider the $N+1$ dimensional perimeter

$$P_{I\times D}(Q) = ||\nabla_{t,x}\zeta||_{M^1(I\times D)}$$

$$= \int_0^1 \int_{\Gamma_t} \sqrt{1+v^2}\, d\Gamma_t \ dt \ \le \int_0^1 P_D(\Omega_t)\, dt$$
$$+ \int_0^1 \int_{\Gamma_t} |v(t)|\, d\Gamma_t\, dt \quad \text{(in smooth case)}$$

Consider also the fact that

$$< \frac{\partial}{\partial t}\zeta, \ g >_{\mathcal{M}(I\times D)\times C^0_{comp}(I\times D)}$$

$$= \int_0^1 \int_{\Gamma_t} v\, g\, d\Gamma_t\, dt = \int_0^1 \int_{\Omega_t} div(g\, V\,) dx dt$$

As

$$\int_0^1 \int_{\Gamma_t} |v|\, d\Gamma_t\, dt \ = \ ||\frac{\partial}{\partial t}\zeta||_{M^1(I\times D)},$$
$$P_D(\Omega_t) \ = \ ||\nabla_x \zeta(t)||_{M^1(D,R^N)}$$

Then we have :

$$||\nabla_{t,x}\zeta||_{M^1(I\times D)} \le \ ||\frac{\partial}{\partial t}\zeta||_{M^1(I\times D)}$$
$$+ \int_0^1 ||\nabla_x \zeta(t)||_{M^1(D,R^N)}\ dt \qquad (1)$$

We shall consider the weak closure of such smooth tubes ζ and verify that the estimate 1 still hold true on the closure:

Proposition 1.1 *let ζ_n be a sequence of smooth tubes such that*

$$||\frac{\partial}{\partial t}\zeta_n||_{M^1(I\times D)}$$
$$+ \int_0^1 ||\nabla_x \zeta_n(t)||_{M^1(D,R^N)}\ dt \le M \qquad (2)$$

Then there exists a subsequence (still denoted ζ_n) and ζ such that $\zeta_n \to \zeta$ stongly in $L^1(I \times D)$ (so that $\zeta = \zeta^2$) and we have :

$$||\nabla_{t,x}\zeta||_{M^1(I\times D)} \leq lim\,inf\,||\frac{\partial}{\partial t}\zeta_n||_{M^1(I\times D)}$$

$$+\int_0^1 ||\nabla_x \zeta_n(t)||_{M^1(D,R^N)}\,dt \qquad (3)$$

Corollary 1.2 *Let $\zeta \in L^1(I, BV(D)) \cap W^{1,1}(I, M^1(D))$ then $\zeta \in C^0(I, L^1(D))$ and a.e. $t \in I$, $\zeta(t) = \chi_{\Omega_t}$ with $P_D(\Omega_t) < \infty$ and $t \to P_D(\Omega_t)$ is l.s.c.*

The weak closures $\mathcal{H}_p^{c,*}$ and $\mathcal{H}_\theta^{c,*}$ of $\mathcal{H}^k$:

$$\mathcal{H}_p^{c,*} = \{\zeta = \zeta^2 \in \mathcal{H}^c \cap \mathcal{H}^*,\ s.t.$$

$$,\ \exists \zeta_n \in \mathcal{H}^k,\ \zeta_n \to \zeta \text{ in } L^1(I\times D)\ ,$$

$$\nabla_{t,x}\zeta_n \to \nabla_{t,x}\zeta \text{ (wealky in) } M^1(I\times D),$$

$$\text{with } \frac{\partial}{\partial t}(\zeta_n - \zeta) \to 0\ \ weakly\ in\ L^P(I, M^1(D))\ \}$$

$$\mathcal{H}_\theta^{c,*} = \{\zeta = \zeta^2 \in \mathcal{H}^c \cap \mathcal{H}^*,\ s.t.$$

$$\exists \zeta_n \in \mathcal{H}^k,\ \zeta_n \to \zeta \text{ in } L^1(I\times D)\ ,$$

$$\nabla_{t,x}\zeta_n \to \nabla_{t,x}\zeta \text{ (wealky in) } M^1(I\times D),$$

$$\text{with } a.e.t \in I,\ ||\frac{\partial}{\partial t}\zeta_n(t)||_{M^1(D)} \leq \theta(t)\ \}$$

2. A COMPLETE QUASI-METRIC SPACE

$$\mathcal{H}_p^{c,*} := \{\ \zeta \in \mathcal{H}^c \cap W^{1,1}(I, M^1(D)),$$

$$\frac{\partial}{\partial t}\zeta \in L^p(I, M^1(D))\ \} \qquad (4)$$

when the moving boundary is smooth :

$$||\frac{\partial}{\partial t}\zeta||_{L^1(I,M^1(D,R^N))} = \int_0^1 ||v(t)||_{M^1(\partial\Omega_t)}dt$$

We consider the variationnal problem

$$\mathcal{T}_p^{c,*}(\Omega_1,\Omega_2) = \{\ \zeta \in \bar{\mathcal{T}}(\Omega_1,\Omega_2) \cap \mathcal{H}_p^{c,*}\ \}\ (5)$$

$$= \{\ \zeta \in \mathcal{H}_p^{c,*}\ s.t.\ \zeta(0) = \chi_{\Omega_1},\ \zeta(1) = \chi_{\Omega_2}\ \}$$

$$j = Inf_{\{\zeta \in \mathcal{T}_p^{c,*}(\Omega_1,\Omega_2)\}}\ \ \{\ ||\frac{\partial}{\partial t}\zeta||_{L^1(I,M^1(D,R^N))}$$

$$+ ||p||_{L^1(I)}\ \} \qquad (6)$$

Proposition 2.1 *Let $p > 1$, there exists (at least one) tube ζ in $\mathcal{T}_p^{c,*}(\Omega_1,\Omega_2) \subset \mathcal{H}_p^{c,*}$ verifying the minimum in the variational problems 6*

The positive number j cannot be zero, $j > 0$, so that j fails to be a distance on the family $\mathcal{O}^*_{\Omega_0,p}$

$$= \{\Omega\, s.t.\, \exists\, \zeta \in \mathcal{H}_p^{c,*}, \chi_\Omega = \zeta(1), \zeta(0) = \chi_{\Omega_0}\}$$

Incorporate the perimeter integral as a constraint in the family:

for given $M > 0$ consider $\mathcal{O}^{*,P_M}_{\Omega_0,p}$

$$= \{\ \Omega \in \mathcal{O}^*_{\Omega_0,p}\ s.t.\ \chi_\Omega = \zeta(1),\ \zeta \in \mathcal{T}^{c,*}(\Omega_0,\Omega),$$

$$\int_0^1 ||\nabla\zeta(t)||_{M^1(D)}\,dt\ \leq\ M\ \ \}$$

Notice that for two element $\Omega_i \in \mathcal{O}^{*,p_M}_p$ there exists connecting tubes verifying the perimeter constraint:

Lemma 2.2 *Let $\Omega_i \in \mathcal{O}^*_{\Omega_0,p},\ i = 1,2$. Then the set $\bar{\mathcal{T}}(\Omega_1,\Omega_2) \cap \mathcal{O}^*_{\Omega_0,p}$ is non empty*

Theorem 2.3 *Let $M > ||\nabla\chi_{\Omega_0}||_{M^1(D,R^N)}$.Let $p = 1$, equipped with $\bar{\delta}$ the family $\mathcal{O}^{*,p_M} \subset \mathcal{P}_D$ is a metric space.*

Let $p > 1$, equipped with $\bar{\delta}$ the family $\mathcal{O}^{*,p_M} \subset \mathcal{P}_D$ is a complete quasi-metric space, in the sense that the triangle inequality is replaced by the following one :

$$\bar{\delta}_p(\Omega_1,\Omega_3)\ \leq\ 2^{p-1}\{\ \bar{\delta}_p(\Omega_1,\Omega_2) + \bar{\delta}_p(\Omega_2,\Omega_3)\ \} \qquad (7)$$

In a full paper (27) we discuss the possibility to introduce the *curvature* term p' in the metric .

3. FULLY EULERIAN METRIC SPACE

As soon as the speed vector field V verifies some BV properties ($V \in L^2(I, BV(D)^N)$) (24; 15), there is a unique tube associated to V, then we do have an application $V \to \zeta_V$ and with such regularity on V we can revisit the complete metric d being completely delivered of the non differential perimeter and curature terms that we were obliged to introduce in order to apply the compacity theorems. From the tube analysis we consider several interesting choices for the spacial regularity of the speed vector field (together with its divergence field). Let

$$\mathcal{E}^{1,1} = \{V \in L^1(I\times D, R^N),$$

$$divV \in L^1(D), V.n_D,\ W^{-1.1}(\partial D\ \},$$

and let E be by closed subspace in $BV(D) \cap \mathcal{E}^{1,1}$ such that any element $V \in E$ verifies the required assumptions. A first example is, when working with prescribed volume for the moving domain,

$$E_0 = \{\ V \in BV(D, R^N) \cap \mathcal{E}^{1,1},$$

$$s.t.\ divV = 0\ a.e.\ (t,x) \in I \times D\ \ \}$$

V be a free divergence vector field with $divV = 0,\ ,\ V \in L^1(I, E_0))$, where $E = BV(D, R^N)$ or any closed subspace (for example $E = \{\ V \in H_0^1(D, R^N),\ s.t.\ \ divV = 0\ \}\ )$. An *obvious* metric is to consider the set

$$\mathcal{V}(\Omega_1, \Omega_2) = \{\ V \in \mathcal{E}^{1,1}\ s.t. V,\ divV\ \in L^p(I, E_0),$$

$$s.t.\ \zeta_0 = \chi_{\Omega_1},\ \zeta(1) = \chi_{\Omega_2}\ \}$$

$$\delta_{E_0}(\Omega_1, \Omega_2) = Inf_{V\in\mathcal{V}(\Omega_1,\Omega_2)}\ \int_0^1 ||V(t)||_{E_0}\, dt \tag{8}$$

As V is divergence free the previous boundedness assumption on the divergence are verified and to each V a tube ζ_V is associated trough the convection. Then we get the

Proposition 3.1 *Let E be any subspace of $BV(D, R^N) \cap \mathcal{E}^{1,1}$ such that any element V satisfies assumptions of theorem 2,12 of (25), for example $E = E_0$. Then equipped with δ_E the family $\mathcal{O}^E_{\Omega_0}$ is metric space.*

$$p > 1, d_{E_0}(\Omega_1, \Omega_2) = Inf_{V\in\mathcal{V}(\Omega_1,\Omega_2)}\quad ||V||_{L^p(I,E_0)}$$

$$+\ ||\frac{\partial}{\partial t}V||_{L^1(I,M^1(D,R^N)} \tag{9}$$

Theorem 3.2 *Let E be any subspace of $BV(D, R^N) \cap \mathcal{E}^{1,1}$, such that any element V whose divergence satisfies assumptions of theorem 2,12 of (25). Then equipped with d_E the family $\mathcal{O}^E_{\Omega_0}$ is a complete quasi- metric space.*

3.1. Geodesic characterizarion via transverse field Z

That metric can be improved as a complete metric by adding the perimeter terms . Then the transverse tube perturbation will applies. In that setting we are concerned with vector fileds $Z(s, t, x) \in R^N$ such that $Z(s, 0, x) = Z(s, 1, x) = 0$ so that the extrimities of the pertubed tube are preserved. The previous study for the transverse field implies that for given such a vector filed Z, with $div_x Z(s, t, x) = 0$ we get the admissible perturbation of the field V in the following form $V + sW(s, t, x)$ with

$$W(s, t, x) = \frac{\partial}{\partial t}Z(s, t, x)\ + [Z,\ V]$$

more precisely define the Lipschitz-continuous connecting set

$$\mathcal{V}^{1,\infty}(\Omega_1, \Omega_2) = \{\ V \in L^1(I, W^{1,\infty}$$

$$\cap\mathcal{E}^{1,1},\ s.t.\ \ \zeta_V\ \in \bar{\mathcal{T}}(\Omega_1, \Omega_2)\ \ \}$$

And the set of smooth transverse vector fields:

$$\mathcal{Z} = \{\quad Z(t, x) \in C^\infty_{comp}(I \times D, R^N)\quad \}$$

(Notice that such Z verifies $Z(0, .) = Z(1, .) = 0$ on D)

Proposition 3.3 *Let $V \in \mathcal{V}(\Omega_1, \Omega_2)$ and $Z(t, x) \in \mathcal{Z}$. The Transformation $\mathcal{T} = T_s(Z)oT_t(V)$ maps $\Omega_t(V)$ onto $\Omega^s_t := T_s(Z)(\Omega_t(V))$ so that*

$$\forall s,\ \forall Z,\ V^s(t, x) = \frac{\partial}{\partial t}\mathcal{T}o\,\mathcal{T}^{-1}$$

$$= (\frac{\partial}{\partial t}T_s(Z(t)) + DT_s(Z(t)).V(t)\ )oT_s(Z(t))^{-1}$$

$$\in\ \mathcal{V}^{1,\infty}(\Omega_1, \Omega_2)$$

Lemma 3.4

$$\frac{\partial}{\partial s}V^s(t, x)|_{s=0}\ =\ \frac{\partial}{\partial t}Z(t) + [Z(t),\ V(t)] \tag{10}$$

Corollary 3.5 *Consider a functional $\mathcal{J}(V) = j(\zeta_V)$ and let $\bar{V}$ be a minimizing element of $\mathcal{J}$ on $\mathcal{V}(\Omega_1, \Omega_2)$ then we have*

$$\forall Z \in \mathcal{Z},\ \frac{\partial}{\partial s}\mathcal{J}(\bar{V}^s)_{s=0}$$

$$= J'(\bar{V};\ (\frac{\partial}{\partial s}V^s)_{s=0})$$

$$= \mathcal{J}'(\bar{V};\ \frac{\partial}{\partial t}Z(t) + [Z(t),\ V(t)]\ \ )\ \geq\ 0 \tag{11}$$

That variational principle extends to vector field $V \in E$ for which the flow mapping $T_t(V)$ is poorly defined. The element $\zeta_V \in \mathcal{H}^c$ is uniquely defined. For any $Z \in \mathcal{Z}$ the perturbed $\zeta^s_V := \zeta_V oT_s(Z)^{-1} \in \bar{\mathcal{T}}(\Omega_1, \Omega_2)$ on the other hand the following result is easily verified

Proposition 3.6 $\zeta_V^s = \zeta_{V^s}$ *with*

$$V^s(t,.) := -DT_s^{-1}(-Z(t)).(\,V(t)oT_s(Z(t))^{-1})$$
$$-\frac{\partial}{\partial t}T_s(-Z(t))\;)$$

In other words:

$$\frac{\partial}{\partial t}\zeta \,+\nabla\zeta.V = 0 \;\; implies$$

$$\frac{\partial}{\partial t}(\zeta oT_s(Z(t))^{-1}) +\nabla(\zeta oT_s(Z(t))^{-1}).V^s \,=\, 0$$

It can also be verified that the expression 10 for the derivative of the field still holds true so that the variational principle (11) is valid for any functional $\mathcal{J}$ minimized over the lipschitzian connecting family $\mathcal{V}^{1,\infty}(\Omega_1,\Omega_2)$. And more generally, without assuming V in E we have :

Proposition 3.7 *Let* $(\zeta,V)\in\mathcal{T}^{p,q}(\Omega_1,\Omega_2)$*, then for all* $s>0$ *and* $Z\in\mathcal{Z}$ *we have :*

$$(\zeta oT_s(Z)^{-1},\,V^s)\,\in\,\mathcal{T}^{p,q}(\Omega_1,\Omega_2)$$

In order to get a differentiable metric we could consider

$$\tilde{d}(\Omega_1,\Omega_2)\,=\,Inf_{V\in\mathcal{V}(\Omega_1,\Omega_2)}$$
$$\int_0^1 (\;||V(t)||_{H_0^1\cap E_0}\,+\,||\frac{\partial}{\partial t}V||_{L^2(D)}\;)\,dt$$

equipped with $\tilde{d}$, $\mathcal{O}_{\Omega_0}$ would be complete metric space but $\tilde{d}$ fails to be a metric because of the triangle axiom The advantage is that now the associated functional is differentiable with repect to V then we can apply the previous variational principle with transverse vector field Z. Let $\bar{V}$ be a minimizer in $\mathcal{V}(\Omega_1,\Omega_2)$ for $\tilde{d}(\Omega_1,\Omega_2)$. Then $\forall Z\in\mathcal{Z}$ we have

$$\int_0^1 \{\,||V(t)||^{-1}\,< V(t),Z_t+[Z,V]>$$
$$+|V'(t)|^{-1}\,((V'(t)\,(Z_t+Z,V)'\,))\;\;\}dt = 0$$

Where $<,>$ is the $H_0^1(D,R^N)$ inner product while $((,))$ is the $L^2(D,R^N)$ one. In order to recover a differentiable complete metric we introduce again the constraint on the perimeter as in the begining and set

$$\delta_{H^1}(\Omega_1,\Omega_2)\,=\,Inf_{V\in\mathcal{V}(\Omega_1,\Omega_2)}$$

$$\int_0^1 \;||V(t)||_{H_0^1\cap E_0}\,dt \tag{12}$$

The optimality condition is :$\forall Z\in\mathcal{Z}$

$$s.t.\;\int_0^1\int_{\Gamma_t} H(t)\;<Z(t),n_t>\;d\Gamma_t\,dt\,=\,0,$$
$$\int_0^1 \;||V(t)||^{-1}\;< V(t),Z_t+[Z,V]\;>\,dt = 0$$

4. QUASI-METRIC BY LEVEL SET FORMULATION FOR APPLICATIONS

Let $p>1$ and Ω_i, $i=1,2$ be two arbitrary mesurable subsets in D. Let

$$K(\Omega_1,\Omega_2) = \{\phi\in L^2(I,H^1(D))\cap W^{1,1}(I,L^2(D)),$$
$$\frac{\partial}{\partial t}\phi\in L^p(I,L^2(D)),$$
$$\Omega_1=\{\Phi(0,.)>0\},\;\Omega_2\;=\{\,\Phi(1,.)>0\,\}$$

Notice that $K(\Omega_1,\Omega_2)\subset\,C^0(\bar{I},L^2(D))$, we set

$$d_{LS,p} = (\Omega_1,\Omega_2) := Inf_{\{\phi\in K(\Omega_1,\Omega_2)\}}$$
$$\int_0^1 (\;\alpha\;||\phi(t)||^2_{H^1(D)}\,+\,||\frac{\partial}{\partial t}\phi(t)||^p_{L^2(D)}\;)\,dt$$

Theorem 4.1 *Let* $1<p\leq 2$*, equipped with* $d_{LS,p}$ *the family of mesurable subsets in* D *is a complete quasi-metric space.*

5. EULER EQUATION FOR GEODESICS

$$\exists c(t),\,P\;\;\;s.t.\;\frac{\partial}{\partial t}(||V(t)||^{p-2}\,V(t))$$
$$+\,||V(t)||^{p-2}\,(\;DV(t).V\,+\,D^*V.V(t)\,)$$
$$=\nabla P + c\;\chi_{\Gamma_t}\,div_{\Gamma_t}(n_t)\;n_t.$$

That is,

$$(p-2)||V||^{p-4}((V,\frac{\partial}{\partial t}V))V$$
$$+\,||V(t)||^{p-2}\,(\frac{\partial}{\partial t}V\,+\,DV(t).V\,+\,D^*V.V(t)\,)$$
$$= c\;\chi_{\Gamma_t}\,div_{\Gamma_t}(n_t)\;n_t, \tag{13}$$

which can be written as (with the notations $\bar{V} = ||V||^{-1}\,V,\;\Pi = P-1/2|V|^2$) :

$$divV = 0,$$
$$\frac{\partial}{\partial t}V+(p-2)((\frac{\partial}{\partial t}V,\,\bar{V}))\,\bar{V}$$
$$=\,DV.V\,=\nabla\Pi{+}c(t)||V||^{2-p}\;\chi_{\Gamma_t}\,div_{\Gamma_t}(n_t)\;n_t \tag{14}$$

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