A Factorization Method for a Singular Perturbation Problem

J. Henry, B. Louro, A. Ramos

INRIA-Futurs; MAB Université Bordeaux 1; 351, cours de la libération, 33405 TALENCE, France; jacques.henry@inria.fr
Departamento de Matemática; Faculdade de Ciências e Tecnologia; Universidade Nova de Lisboa; 2829-516 Caparica, Portugal; bj@fct.unl.pt
Departamento de Matemática Aplicada; Universidad Complutense de Madrid; Plaza de Ciencias 3; 28040 Madrid, Spain; angel@mat.ucm.es

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1. INTRODUCTION

We want to solve problems on cylinders $\Omega^\varepsilon = [0, a] \times O^\varepsilon$ in $\mathbb{R}^d$, with $O^\varepsilon = \varepsilon O$ and $O$ a bounded open set in $\mathbb{R}^{d-1}$ ($d = 2$ or $3$ in real applications). Parameter $\varepsilon$ denotes that the section $O^\varepsilon$ is much smaller than the length of the axis $a$.

We denote $\Gamma^e_\varepsilon = \{s\} \times O^\varepsilon$, the lateral boundary of the cylinder $\Sigma^e = [0, a] \times \partial O^\varepsilon$ and a general point $(x_1^\varepsilon, x_2^\varepsilon, \ldots, x_n^\varepsilon) \in \Omega^\varepsilon$ is also denoted by $(x^\varepsilon, y^\varepsilon)$, where $x^\varepsilon = x_1^\varepsilon$ and $y^\varepsilon$ denotes the independent variables $(x_2^\varepsilon, \ldots, x_n^\varepsilon)$. Let $f \in L^2(\Omega^\varepsilon)$, $u_0 \in H^{1/2}(O^\varepsilon)$, $u_a \in H^{1/2}(O^\varepsilon)'$ and $\lambda$ be a positive constant. The problem we want to solve is

\[
\begin{cases}
-\Delta u + \lambda u = f & \text{in } \Omega^\varepsilon, \\
\partial u/\partial \nu = 0 & \text{on } \Sigma^e, \\
u = u_0 & \text{on } \Gamma_0^\varepsilon, \\
\partial u/\partial x = u_a & \text{on } \Gamma_a^\varepsilon,
\end{cases}
\]

where $\nu$ represents the outward normal vector on the boundary.

The aim of this work is to study the asymptotic behavior of the solution when $\varepsilon \to 0$. We do first a change of variable so that the new domain is the same for all $\varepsilon > 0$. We consider the domain given by the cylinder $\Omega = [0, a] \times O$ and the change of variable $(x^\varepsilon, y^\varepsilon) = (x, \varepsilon y)$. Therefore, in the new variables $(x, y)$ (1) can be re-written equivalently by

\[
\begin{cases}
-\frac{\partial^2 u}{\partial x^2} - \frac{1}{\varepsilon^2} \Delta_y u + \lambda u = f & \text{in } \Omega, \\
\frac{\partial u}{\partial \nu} = 0 & \text{on } \Sigma, \\
u = u_0 & \text{on } \Gamma_0, \\
\frac{\partial u}{\partial x} = u_a & \text{on } \Gamma_a,
\end{cases}
\]

2. FACTORIZATION BY INVARIANT EMBEDDING

Factorizing problem (2) by invariant embedding techniques as in Henry et.al (2004b) and Henry et.al (2004a), we arrive to the uncoupled system:

\[
\begin{cases}
-\frac{\partial^2 Q}{\partial x^2} + Q^2 = \lambda I + \frac{1}{\varepsilon^2} A, \\
Q(a) = 0
\end{cases}
\]

\[
\begin{cases}
-\frac{\partial w}{\partial x} + Qw = -f, \\
w(a) = -u_a
\end{cases}
\]

\[
\begin{cases}
\frac{\partial u}{\partial x} + Qu = -w, \\
u(0) = u_0
\end{cases}
\]

where $(Ah, \varphi) = (\nabla_y h, \nabla_y \varphi) \forall h, \varphi \in H^1(O)$. $A$ is the abstract operator corresponding to Neumann boundary conditions for the laplacian.

3. SINGULAR PERTURBATIONS

Let $e_0, e_1, \ldots, e_n, \ldots$ be an orthonormal basis (with respect to the $L^2(O)$ norm) of eigenvectors of $A$ in the following sens
Let $\mathcal{W}^r$ be the space defined by
\[
u = \sum_{i=1}^{+\infty} u_i \epsilon_i \in \mathcal{W}^r \iff \|u\|_{\mathcal{W}^r}^2 = \sum_{i=1}^{+\infty} \lambda_i^r |u_i|^2 < +\infty
\]

We consider the formal development
\[
Q = \varepsilon^{-1} Q_{-1} + Q_0 + \varepsilon Q_1 + \varepsilon^2 Q_2 + \cdots,
\]
with $Q_i$, $i = -1, 0, 1, \cdots$, self-adjoint and non-negative. We also decompose $Q_i$, for $i \geq 0$, as
\[
Q_i \varphi = \begin{pmatrix} Q_i^{aa} & Q_i^{ab} \\ Q_i^{ba} & Q_i^{bb} \end{pmatrix} \begin{pmatrix} \varphi^a \\ \varphi^b \end{pmatrix},
\]
with
\[
Q_i^{aa} : \mathcal{W}^1 \to L^2(\mathcal{O})/\mathbb{R}
\]
\[
Q_i^{ab} : \mathbb{R} \to L^2(\mathcal{O})/\mathbb{R}
\]
\[
Q_i^{ba} : \mathcal{W}^1 \to \mathbb{R}
\]
\[
Q_i^{bb} : \mathbb{R} \to \mathbb{R}.
\]

We obtain the following results:
\[
Q_{-1} = Q_{-1}^{aa} = A^{1/2},
\]
\[
w_{-1} = 0,
\]
\[
Q_0 = Q_0^{bb}(x) = \sqrt{\lambda} \tanh(\sqrt{\lambda}(a - x)),
\]
\[
w_0(x) = w_0^{bb}(x) = \frac{u_b^{bb}}{\cosh(\sqrt{\lambda}(a - x))} - \frac{f_b^{bb}}{\sqrt{\lambda}} \tanh(\sqrt{\lambda}(a - x)),
\]
\[
Q_1 = Q_1^{aa} = \frac{1}{2} A^{-1/2},
\]
\[
w_1(x) = w_1^{aa}(x) = -A^{-1/2} f^a,
\]
\[
Q_2 = 0,
\]
\[
w_2(x) = w_2^{aa}(x) = -A^{-1} \frac{d}{dx} f^a,
\]
\[
Q_3 = Q_3^{aa} = \frac{\lambda}{2} A^{-3/2},
\]
\[
w_3(x) = w_3^{aa}(x) = -A^{-3/2} \frac{d^2}{dx^2} f^a + \frac{\lambda}{2} A^{-3/2} f^a.
\]

As one can see, all the transverse operators $Q_i^{ab}$ and $Q_i^{ba}$ are zero and all the $Q_i$ only dependent on $y$ except $Q_0$ that only dependent on $x$.

REFERENCES

