Morrey Regularity and Continuity of Minimizers for Asymptotically Convex Integrals

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1. Introduction

I will present some Morrey regularity results for minimizers of functionals with the general form

$$\mathbf{u} \mapsto \int_\Omega g(\mathbf{x}, \mathbf{u}(\mathbf{x}), \boldsymbol{\nabla}\mathbf{u}(\mathbf{x}))\, d\mathbf{x}, \tag{1}$$

where Ω is an open, bounded subset of $\mathbb{R}^n$ and $\mathbf{u} \in W^{1,1}(\Omega;\mathbb{R}^N)$, with $n, N \geq 1$. The primary property that I assume g possesses is that there is a $p \in (1,\infty)$ such that for each $\mathbf{x} \in \mathbb{R}^n$ and each $\mathbf{u} \in \mathbb{R}^N$, the function $\mathbf{F} \mapsto g(\mathbf{x},\mathbf{u},\mathbf{F})$ behaves like $\mathbf{F} \mapsto \|\mathbf{F}\|^p$ whenever $\|\mathbf{F}\|$ is sufficiently large. Integrands with this property are called asymptotically convex.

To make things more precise, let us say that a function $g : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n} \to \mathbb{R}$ is $\mathcal{C}^0$-asymptotically convex if for each $\varepsilon > 0$ and each $(\mathbf{x},\mathbf{u}) \in \Omega \times \mathbb{R}^N$, there exists a $\sigma_\varepsilon(\mathbf{x},\mathbf{u}) < +\infty$ such that

$$\left| g(\mathbf{x},\mathbf{u},\mathbf{F}) - \left(1 + \|\mathbf{F}\|^2\right)^{\frac{p}{2}} \right| < \varepsilon\|\mathbf{F}\|^p, \quad (\mathcal{C}^0\text{-A})$$

whenever $\|\mathbf{F}\| > \sigma_\varepsilon(\mathbf{x},\mathbf{u})$. If $g : \mathbb{R} \to \mathbb{R}$ is given by $g(F) := \left(1 + |F|^2\right)^{\frac{p}{2}} - |F|\chi_{\mathbb{Q}}$, where $\chi_{\mathbb{Q}}$ is the characteristic function for the set of rational numbers, then we see that g is $\mathcal{C}^0$-asymptotically convex with $\sigma_\varepsilon = \varepsilon^{-\frac{1}{p-1}}$, yet g is nowhere convex. Nevertheless, one can show that a $\mathcal{C}^0$-asymptotically convex function does, in some sense, behave like a convex function at infinity. Our regularity results apply to a minimizer, provided one exists, for functionals of the general form (1), provided that g is $\mathcal{C}^0$-asymptotically convex and the function $(\mathbf{x},\mathbf{u}) \mapsto \sigma_\varepsilon(\mathbf{x},\mathbf{u})$ satisfies some growth and regularity conditions.

2. Statement of Result

The statements for the main results are given in terms of a generalized notion of an almost minimizer and are fairly technical, so I present an application which conveys an idea of the content of the main results while reducing the technicalities. In the following, I use $L^{p,\kappa}$ to denote a Morrey space and $\mathscr{L}^{p,\kappa}$ to denote a Companato space.

Theorem 1 *Let $0 \leq \kappa < n$, $0 \leq s < r < +\infty$ and $1 < q < +\infty$ be given. Let $\alpha \in L^{1,\kappa}(\Omega)$ be given. Suppose that there is a $\lambda \geq 0$ such that $h : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n} \to \mathbb{R}$ satisfies*

$$|h(\mathbf{x},\mathbf{u},\mathbf{F})| \leq \alpha(\mathbf{x}) + \lambda\|\mathbf{u}\|^r + \|\mathbf{F}\|^q,$$

for each $(\mathbf{x},\mathbf{u},\mathbf{F}) \in \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n}$. Let $\delta > 0$ and $p > \max\left\{q, \frac{nr}{n+r}\right\}$ be given. Define the functional $J : W^{1,1}(\Omega;\mathbb{R}^N) \to \overline{\mathbb{R}}$ by

$$J[\mathbf{u}] := \int_\Omega \left\{ \delta\left(1 + \|\boldsymbol{\nabla}\mathbf{u}(\mathbf{x})\|^2\right)^{\frac{p}{2}} + h(\mathbf{x},\mathbf{u}(\mathbf{x}),\boldsymbol{\nabla}\mathbf{u}(\mathbf{x})) \right\} d\mathbf{x}.$$

We have the following: If $\mathbf{u} \in W^{1,p}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$ is a local minimizer for J; i.e. $J[\mathbf{u}] \leq J[\mathbf{u}+\boldsymbol{\varphi}]$, for each $\boldsymbol{\varphi} \in W^{1,p}(\Omega;\mathbb{R}^N)$ with $\operatorname{supp}(\boldsymbol{\varphi}) \subset\subset \Omega$; then $\boldsymbol{\nabla}\mathbf{u} \in L^{p,\kappa}_{\mathrm{loc}}(\Omega;\mathbb{R}^{N\times n})$ and $\mathbf{u} \in \mathscr{L}^{p,p+\kappa}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$.

This result actually holds up to the boundary provided that $\partial\Omega$ and the boundary conditions are sufficiently smooth. It also holds for certain variational problems with sufficiently smooth obstacles. It is also possible to allow the coefficient δ to be a continuous function that is uniformly positive in Ω.

3. Conclusion

To conclude, I make a few comments about the implications of the above result to the broader endeavor of establishing a lower-order regularity theory for variational problems. Until recently, results for such a theory have been for the most part unavailable (see (2) for a discussion). As demonstrated by V. Šverák & X. Yan (3), even if an integrand $h \in \mathcal{C}^\infty(\mathbb{R}^{N\times n})$ is strictly convex and has a uniformly bounded Hessian, a minimizer for the functional $\mathbf{u} \mapsto \int_\Omega h(\boldsymbol{\nabla}\mathbf{u}(\mathbf{x}))\,d\mathbf{x}$ can be unbounded at an interior point in Ω. Thus without additional assumptions on h, one can not expect everywhere regularity for a minimizer. In (1), M. Foss & G. Mingione showed that if $h \in \mathcal{C}^0(\Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n})$ is quasiconvex and possesses some additional growth and continuity properties with respect to its third argument, then a minimizer for the functional

$$\mathbf{u} \mapsto \int\limits_\Omega h(\mathbf{x}, \mathbf{u}, \boldsymbol{\nabla}\mathbf{u}(\mathbf{x}))\,d\mathbf{x} \qquad (2)$$

is partially continuous; i.e. continuous on an open subset of Ω with full measure. Since $\mathscr{L}^{p,p+\kappa} \subset \mathcal{C}^{0,1-\frac{n-\kappa}{p}}$ whenever $p+\kappa > n$, Theorem 1 shows that for each $\delta > 0$ minimizers for the functional

$$\mathbf{u} \mapsto \int\limits_\Omega \left\{\delta\left(1 + \|\boldsymbol{\nabla}\mathbf{u}(\mathbf{x})\|^2\right)^{\frac{p}{2}} + h(\mathbf{x}, \mathbf{u}(\mathbf{x}), \boldsymbol{\nabla}\mathbf{u}(\mathbf{x}))\right\} d\mathbf{x} \qquad (3)$$

must be everywhere continuous, provided that $p > \max\left\{q, \frac{nr}{n+r}, n-\kappa\right\}$. This result only requires h to satisfy some very mild growth conditions. Thus $\delta\left(1 + \|\boldsymbol{\nabla}\mathbf{u}(\mathbf{x})\|^2\right)^{\frac{p}{2}}$ serves as a rather robust regularizing term. It would be interesting to discover if it is possible to obtain information about the lower-order regularity of a minimizer for the functional in (2) by approximating it using minimizers for functionals of the form (3) and the regularity provided by Theorem 1.

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