REGULARITY PROPERTIES OF OPTIMAL CONTROL FOR SOME MIXED CONSTRAINED PROBLEMS

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1. INTRODUCTION

In this work we investigate regularity properties of optimal control for problems with mixed state-control constraints. To the best of our knowledge, regularity properties of optimal control for such problems have not been studied previously. We consider on a problem \((P_{eq})\) with equality type constraints:

\[
\begin{aligned}
\text{Minimize } & l(x(1)) \\
& \quad + \frac{1}{2} \int_0^1 (x(t)^*Qx(t) + u(t)^*Ru(t)) dt \\
\text{subject to } & \begin{cases}
\dot{x}(t) = f(t, x(t)) + B(t)u(t) & \text{a.e.} \\
0 = Cx(t) + Eu(t) & \text{a.e.} \\
x(0) \in C_0
\end{cases}
\end{aligned}
\]

and on a problem \((P_{in})\) with inequality type constraints:

\[
\begin{aligned}
\text{Minimize } & l(x(1)) \\
& \quad + \frac{1}{2} \int_0^1 (x(t)^*Qx(t) + u(t)^*Ru(t)) dt \\
\text{subject to } & \begin{cases}
\dot{x}(t) = f(t, x(t)) + B(t)u(t) & \text{a.e.} \\
0 \geq Cx(t) + Eu(t) & \text{a.e.} \\
x(0) \in C_0
\end{cases}
\end{aligned}
\]

The data for these problems comprise state and control variables \(x : [0, 1] \to \mathbb{R}^n\), \(u : [0, 1] \to \mathbb{R}^k\), functions \(l : \mathbb{R}^n \to \mathbb{R}\), \(f : [0, 1] \times \mathbb{R}^n \to \mathbb{R}^n\), \(B : [0, 1] \to \mathcal{M}_{m \times k}\), matrices \(Q \in \mathcal{M}_{n \times n}\), \(R \in \mathcal{M}_{k \times k}\), \(C \in \mathcal{M}_{m \times n}\), \(E \in \mathcal{M}_{m \times k}\) and a closed set \(C_0 \subset \mathbb{R}^n\). Here \(\mathcal{M}_{p \times q}\) is the set of all \(p \times q\) matrices with real entries. We assume that \(m < k\), i.e. that the number of constraints is less than the dimension of the control variable.

An important motivation for the study of regularity of optimal control is that prior knowledge of its regularity properties (such as smoothness or Lipschitz continuity) influences the choice of the most effective approximation scheme for numerical solution of optimal control problems. Regularity of optimal control have previously been studied for problems with state constraints or state constraints and pointwise control constraints by, for example, Galbraith et. al (2003); Hager (1979); Malanowski (1978); Shvartsman et. al (2006); Vinter (2000).

In this work we prove that under mild conditions on the data, optimal control is infinitely differentiable in the problem with equality constraints and Lipschitz continuous in the problem with inequality constraints.

We impose the following hypotheses on the data of \((P_{eq})\) and \((P_{in})\).

- (H1) Function \(l\) is locally Lipschitz continuous and \(f\) and \(B\) are \(C^\infty\) functions.
- (H2) The set \(C_0\) is closed.
- (H3) The matrices \(Q\) and \(R\) are symmetric and \(R\) is positive definite.
- (H4) The matrix \(E\) is of full rank, i.e. \(\det EE^* \neq 0\).

Crucial to our analysis is the following result from linear algebra.

Proposition 1.1 If \(E \in \mathcal{M}_{m \times k}\) with \(m < k\) satisfies (H4) then there exist square non-singular matrices \(S \in \mathcal{M}_{m \times m}\) and \(T \in \mathcal{M}_{k \times k}\) such that

\[SET = [I\ 0],\]
i.e. the left \( m \times m \) block in the latter matrix is the identity matrix, and the remaining entries are zeros.

This result follows easily from the Singular Value Decomposition Theorem (see, for example, Theorem 7.3.5 in Horn et al., 1985).

2. MAIN RESULTS

Our main results are the following:

Theorem 2.1 Assume (H1)-(H4). Then the optimal control \( \bar{u} \) in \((P_{eq})\) is a \( C^\infty \)-function.

Theorem 2.2 Assume (H1)-(H4). Then the optimal control \( \bar{u} \) to \((P_m)\) is Lipschitz continuous.

The idea of the proof of both theorems is to reduce the problem under consideration to a problem without mixed constraints, and then to investigate the implications of the Pontryagin Maximum Principle. We illustrate the aforementioned reduction below.

Let \( (\bar{x}, \bar{u}) \) be an optimal process to \((P_{eq})\). Set

\[
\bar{v} = \begin{bmatrix} \bar{v}_1 \\ \bar{v}_2 \end{bmatrix} := T^{-1}\bar{u},
\]

where \( \bar{v}_1 \in \mathbb{R}^m, \bar{v}_2 \in \mathbb{R}^{k-m} \) and \( T \) is from Proposition 1.1. It can be shown that there exist functions \( \bar{f}, \bar{B} \), matrices \( \bar{Q}, \bar{S} \) and a positive-definite matrix \( \bar{R} \) of corresponding dimensions such that \( (\bar{x}, \bar{v}_2) \) is an optimal process to the problem \((P_1)\):

\[
\begin{align*}
\text{Minimize} & \quad l(x(1)) \\
& + \frac{1}{2} \int_{0}^{1} (x(t)^*Qx(t) + 2x(t)^*\bar{S}v_2(t) \\
& + v_2(t)^*\bar{R}v_2(t)) dt \\
\text{subject to} & \quad \dot{x}(t) = \bar{f}(t, x(t)) + \bar{B}(t)v_2(t) \text{ a.e.} \\
x(0) & \in C_0
\end{align*}
\]

Observe that \((P_1)\) is an optimal control problem without mixed constraints.

Similarly, let \( (\bar{x}, \bar{u}) \) be an optimal process to \((P_m)\). Set

\[
\bar{w}(t) = \begin{bmatrix} -C\bar{x}(t) - E\bar{u}(t) \\ \bar{v}_2(t) \end{bmatrix},
\]

where \( \bar{v}_2 \) is defined in (1). It can be shown that \( (\bar{x}, \bar{w}) \) is an optimal process to problem \((P_2)\):

\[
\begin{align*}
\text{Minimize} & \quad l(x(1)) \\
& + \frac{1}{2} \int_{0}^{1} (x(t)^*\bar{Q}x(t) + 2x(t)^*\bar{S}w(t) \\
& + w(t)^*\bar{R}w(t)) dt \\
\text{subject to} & \quad \dot{x}(t) = \bar{f}(t, x(t)) + \bar{B}(t)w(t) \text{ a.e.} \\
x(0) & \in C_0
\end{align*}
\]

with

\[
\Omega = \left\{ (w_1, w_2) \in \mathbb{R}^m \times \mathbb{R}^{k-m} : w_1 \geq 0 \right\}
\]

for some functions \( \bar{f}, \bar{B} \) and matrices \( \bar{Q}, \bar{S} \) and \( \bar{R} \). Note that problem \((P_2)\) does not contain a mixed constraint, but is a problem with a control constraint of a simple structure.

3. CONCLUSIONS

In this paper we establish regularity properties of the optimal control for a simple class of mixed constrained problems. Proposition 1.1 and the main result in (Shvartsman et al., 2006) play an important role in the analysis. We hope to extended Theorem 2.1 and 2.2 to cover more general problems.

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