Curvature of optimal control: Deformation of classical planar systems

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1. Introduction

Consider the problem of deciding whether a trajectory pair \((u^*(t), x^*(t))\), \(t \in [0, T]\) of a generally nonlinear system \(\dot{x} = F(x, u), x \in M^n\) is a time-optimal solution connecting the endpoints \(x(0)\) and \(x(T)\), or whether the system is locally controllable about this trajectory. The classical approach analyzes the endpoint map \(u \mapsto x(T, u)\) (for fixed \(T\) and \(x(0)\)) and determine whether or not it is locally an open map. The Pontryagin Maximum Principle and high-order open-mapping theorems provide necessary conditions for a trajectory-control-pair to be optimal. Sufficient conditions for optimality (and necessary conditions for nonlinear controllability) are harder to obtain. Like the Legendre-Clebsch condition, they generally take the form of tests of definiteness for second order derivatives. Recently Agrachev introduced an attractive alternative by developing a notion of curvature of optimal control that generalizes classical Gauss (and Ricci) curvatures. That theory has been developed for systems whose controls take values in a circle or sphere \(u \in \mathbb{R}^{n-1}\).

We present initial studies of how this notion of curvature provides insight into the limiting case when the circles become degenerate ellipses in the form of closed intervals or lower dimensional cubes. Of particular interest are well studied accessible, but uncontrollable, nonlinear systems, and systems that exhibit conjugate points. We study how the curvature and conjugate points vary when the set of controlled velocities \(S^1 = \{(u_1, u_2): u_1^2 + u_2^2 = 1\}\) is continuously deformed into the interval \(I = [-1, 1]\). For computational reasons we implement this by adding the parameter \(\varepsilon\) into the controlled vector field as follows, and leaving the set of control values \(U = S^1\) the same.

\[
\begin{aligned}
\dot{x}_1 &= f_1(x_1, x_2) + u_1 \\
\dot{x}_2 &= f_2(x_1, x_2) + \varepsilon u_2 \quad u_1^2 + u_2^2 = 1
\end{aligned}
\]  

(1)

Of particular interest are deformations of the well-understood systems (when \(\varepsilon = 0\))

\[
\begin{aligned}
\dot{x}_1 &= u_1 \\
\dot{x}_2 &= x_1^m + \varepsilon u_2
\end{aligned}
\]  

(2)

and

\[
\begin{aligned}
\dot{x}_1 &= -x_2 + u_1 \\
\dot{x}_2 &= x_1 + \varepsilon u_2
\end{aligned}
\]  

(3)

with \(|u_1| \leq 1\).

We are interested in how their properties arise as limits of deformations of the corresponding systems of the form (1). The first family of systems is small-time locally controllable if and only if \(m\) is odd. If \(m\) is even, the reachable sets exhibit well-known fold-overs with consequent appearance of conjugate points.

2. Curvature of optimal control

Unlike the classical Gauss curvature, Agrachev’s curvature is not a function on the state-space, but rather on the cotangent bundle over the state-space. In the case of planar systems, the theory is formulated via a distinguished vertical vector field \(v\) on the cotangent bundle which is characterized by the identity \(L_v^2 s = -s + bL_v s\) where \(s = p_1 dx_1 + p_2 dx_2\) is the tautological one-form on \(T^*\mathbb{R}^2\) restricted to the level surface \(\mathcal{H}\) of the Hamiltonian (and \(L\) denotes Lie

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derivative). Together with the Hamiltonian field $\vec{h}$ and their Lie bracket one obtains a moving frame

$$V_1 = v, \quad V_2 = [v, \vec{h}], \quad V_3 = \vec{h}$$

on the level surface $H^{-1}(1) \subseteq T^*\mathbb{R}^2$. One readily verifies that the Lie derivatives of this frame satisfy

$$[\vec{h}, V_1] = -V_2, \quad [\vec{h}, V_2] = \kappa V_1, \quad [\vec{h}, V_3] = 0$$

where $\kappa$ is a scalar function on $H$ and is called the curvature of the control system (1). Writing the Jacobi equation along an extremal $(x_t, p_t)$ in terms of this moving frame one obtains the time-varying linear differential equation

$$\ddot{y} + \kappa_t y = 0, \quad y(0) = y(t_c) = 0.$$ 

which has no nontrivial solutions when $\kappa \leq 0$. In the case on not necessarily negative curvature, standard integral estimates yield lower bounds on the first positive conjugate time $t_c$.

Notable results for very specific classes of systems were obtained by Serres (4) who studied Zermelo’s navigation problem, basically the undeformed ($\varepsilon = 1$) system (1). Recent work by Agrachev et. al. (2) extended the theory to higher dimensional systems. Complementary to this is recent work by Chitour and Sigalotti, who investigate the Dubins’ car on curved surfaces (3; 5).

3. Deformations and curvature

While most pertinent literature (1; 2; 3; 4; 5), is concerned with the further theoretical development, a main thrust of our work is to explore the boundaries of what is computationally feasible with current technology, suiting a combination of symbolic and numeric engines.

While already in the undeformed case the formula for the curvature in coordinates fills a whole page, in the case of deformed control sets, the formulae become much too large to be reproduced here. One starts with the Hamiltonian vector field in polar coordinates Next we compute the distinguished vertical vector field $v$, and the iterated Lie bracket $[[\vec{h}, [\vec{h}, v]]$ from which we then obtain both formulae for the curvature, now depending on the deformation parameter $\varepsilon \in [0, 1]$, and numerical simulations of the time evolution of the co-state vector along extremals, as well their projections onto the state-space, as illustrated in figures 1 and 2.

REFERENCES