Extended Auxiliary Problem Principle using Bregman distances

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1. Introduction

Let \((V, \| \cdot \|)\) be a Hilbert space with the topological dual \(V'\) and the duality pairing \(\langle \cdot, \cdot \rangle\) between \(V\) and \(V'\). The variational inequality (VI)

\[
\text{find } u^* \in K \text{ and } q^* \in Q(u^*): \\
\langle F(u^*) + q^*, u - u^* \rangle \geq 0 \quad \forall u \in K
\] (1)

is considered, assuming that \(K \subset V\) is a convex closed set, \(Q : V \to 2^{V'}\) is a maximal monotone operator and \(F : K \to V'\) is a weakly continuous operator with certain monotonicity properties.

In the sequel we denote by \(\{K^k\}\) a family of convex closed sets, approximating \(K, K^k \subset V\); and by \(\{Q^k\}\) a family of operators, approximating \(Q\). Usually, it is supposed that \(Q^k\) is maximal monotone, or that

\[
Q \subset Q^k \subset Q_{e,k},
\]

\(Q_e\) means the \(e\)-enlargement of \(Q\).

In proximal point methods (PPM) and the Auxiliary Problem Principle (APP), a regularizing functional \(h\) of Bregman type with zone \(S\) is used, where

\[
S := V, \quad h : u \mapsto \frac{1}{2}\|u\|^2.
\]

In this paper we use a regularizing functional \(h\) of Bregman type with zone \(S = \text{int} \, K\) and consider the following general scheme for solving VI (1):

At step \(k + 1\), having a current iterate \(u^k\) \((u^1 \in K \cap S\) is arbitrarily chosen\)
the point \(u^{k+1}\) is calculated by solving the problem

\[
\left( P^k_{\delta} \right) \quad \text{find } u^{k+1} \in K^k \cap S, q^k \in Q^k(u^{k+1}): \\
\langle F(u^k) + q^k + L^k(u^{k+1}) - L^k(u^k) \\
+ \chi_k (\nabla h(u^{k+1}) - \nabla h(u^k)), u - u^{k+1} \rangle \\
\geq -\delta_k\|u - u^{k+1}\| \quad \forall u \in K^k \cap S.
\] (2)

Here \(\{\delta_k\}\) is a non-negative sequence with \(\lim_{k \to \infty} \delta_k = 0\), whereas \(0 < \chi_k \leq \tilde{\chi} < \infty\).

The main advantage consists in the "interior point effect" of this approach, i.e., \((P^k_{\delta})\) is in fact an unconstrained problem.

2. Set and operator approximation

In the literature, in different regularization methods when approximation of the set \(K\) is included, usually it is supposed that \(\{K^k\}\) converges to \(K\) "sufficiently" fast in the Hausdorff or Mosco sense, for example:

\[
dist_H(K^k, K) \leq c_{\varphi_k}, \quad \sum \frac{\varphi_k}{\chi_k} < \infty.
\]

However, this type of assumptions is not very realistic when dealing with VI's in Mathematical Physics. Indeed, constructing \(\{K^k\}\), \(K^k = K_{h^k}\), by means of the FEM on a sequence of triangulations with parameter \(h_k \to 0\), as well as by related FDM, we meet the following typical situation:

(i) for \(v \in K\) and \(v^k := \arg \min_{z \in K^k} \|v - z\|\) it holds

\[
\lim_{k \to \infty} \|v - v^k\| = 0;
\]

(ii) for \(v \in U^*\) (solution set) it holds

\[
\|v - v^k\| \leq c(v) h_k^{\beta_1}, \quad \beta_1 > 0;
\]

(iii) for a bounded sequence \(\{w^k\}\), \(w^k \in K^k\), the estimate

\[
\min_{v \in K} \|v - w^k\| \leq c h_k^{\beta_2}, \quad \beta_2 > 0
\]

is valid (of course, \(c = 0\) fits the case \(K^k \subset K\), but this inclusion is not guaranteed, in general).


Thus, because of the weak property (i), FEM cannot provide the required Hausdorff or Mosco approximation of $K$.

Considering (i)-(iii) as conditions, together with
\[ \sum \frac{\varphi_k}{\chi_k} < \infty, \quad (3) \]
where $\varphi_k := \max\{h^1_k, h^2_k\}$, $\sum \frac{\delta_k}{\chi_k} < \infty$
we deal with quite different requirements on the type of approximation (cf. (3)).

An approximation of $Q$ by means of smoothing procedures or the use of the $\epsilon$-enlargement concept will be discussed, too (cf. (4)).

3. Bregman-function-based methods

To our knowledge, Bregman functions $h$ with zone $S \neq V$ have not been used in connection with APP.

In different variants of APP the operator $\nabla h$ is supposed to be Lipschitz continuous on $K$ or on some set $K \supset K$. This excludes the use of Bregman-like functions with zone $S \not\supset K$, and in particular with $S \subset K$.

In fact, Bregman functions with zone $S \subset K$ provide a full "interior point effect", i.e. with a certain precaution the auxiliary problems can be treated as unconstrained ones.

Now, we consider scheme (2), allowing $S \subset K$, $S := \text{int} K$. Conditions on $h$ require that
\[ S \cap D(Q) \cap K^k \neq \emptyset, \quad S \cap U^* \neq \emptyset. \]

In order to use our convergence analysis in this case, an approximation of $K$ has to be inserted into the algorithm for solving the subproblems. I.e., the subproblems are considered with $K^k := K$ and within the process of their solution by an appropriate method the approximation of $K$ is realized.

However, in general some additional assumptions on the operator $Q$ are needed, even for the exact PPM with strongly convex $h$. If $Q$ is not symmetric, the paramonotonicity and pseudomonotonicity (in the sense of Brezis-Lions) of $Q$ are supposed. In case $V := \mathbb{R}^n$, Solodov/Svaiter (6) have shown that the pseudo-monotonicity requirement can be omitted, but their arguments are finite-dimensional in essence.

The convergence of the extended APP in form $(P^K)$ with Bregman function $h$ is proved under certain assumptions (cf. (5)).

Up to now, zone - or boundary coercive Bregman functions with zone $\text{int} K$ have been created only for linearly constrained sets $K$ or in the case that $K$ is a ball.

The more general case
\[ K = \{x \in \mathbb{R}^n : g_i(x) \leq 0, \quad i \in I_1 \cup I_2\}, \]
where $I_1 \cup I_2 = I = \{1, \ldots, m\}$, and
\[ g_i (i \in I_1) \text{ affine functions}, \]
\[ g_i (i \in I_2) \text{ convex, } C^1 \text{ functions}, \]
\[ \text{max}\{g_i : i \in I_2\} \text{ is strictly convex on } K, \]
\[ \exists x : g_i(x) < 0, \quad \forall i \in I, \]
allows us to use a special class of Bregman functions with zone $\text{int} K$ (see (5)):
\[ h(x) = \sum_{i=1}^{m} \varphi(g_i(x)) + c\|x\|^2, \quad (c > 0). \quad (4) \]

Certain properties and particular realizations of the function $\varphi$ will be discussed.

REFERENCES


