1. INTRODUCTION

The proximal-point-algorithm (PPA), originally introduced by Rockafellar (7), and the auxiliary-problem-principle (APP), which goes back to Cohen (3) are well-known solution methods for variational inequalities. In the last years, extensions of these methods have been considered e.g. by Kaplan/Tichatschke (5), Eckstein (4), Censor/Iusem/Cenios (2) and Solodov/Svaiter (8).

A proximal-auxiliary-problem (PAP) principle, introduced by Rockafellar (7), and the proximal-point-algorithm (PPA), originally introduced by Kaplan/Tichatschke (6) and a general convergence theory was developed. In an extension of this method they use Bregman distances to achieve an interior-point-effect. In our work we replace these Bregman distances by logarithmic-quadratic distance functions which also lead to an interior-point-effect but don’t have the disadvantage of requiring parmonotonicity of the operator of the considered variational inequality. As a consequence, the logarithmic-quadratic PAP (LQPAP) can be used for a broader class of variational inequality problems.

2. LQPAP-METHOD

We suppose that the operator of the given variational inequality is splitted into the sum of a maximal monotone, set-valued operator \( Q \) and a single-valued, continuous operator \( F \) and consider the problem

\[
\text{VI}(F, Q, K):
\]

find \( x^* \in K \) and \( q^*(x^*) \in Q(x^*) \):

\[
\langle F(x^*) + q^*(x^*), x - x^* \rangle \geq 0 \quad \forall \ x \in K,
\]

where \( K \) has to be a polyhedral subset of \( \mathbb{R}^n \), given by

\[
K = \{ x \in \mathbb{R}^n : Ax \leq b \}
\]}

with \( A \in \mathbb{R}^{p \times n} \), \( \text{rank}(A) = n \), \( b \in \mathbb{R}^p \) and \( \text{int}(K) \neq \emptyset \).

Our extension of the APP for solving \( \text{VI}(F, Q, K) \) can be subsumed under the following general iterative scheme:

Starting with \( x^1 \in \text{int}(K) \), at the \( (k+1) \)th step we have a current iterate \( x^k \in \text{int}(K) \) and calculate \( x^{k+1} \) by solving the problem:

\[
(P_{\delta}^k):
\]

find \( x^{k+1} \in K, \ q^k(x^{k+1}) \in Q^k(x^{k+1}) ; \)

\[
\langle F(x^k) + q^k(x^{k+1}) + L^k(x^{k+1}) - L^k(x^k) + \chi_k \nabla \mathcal{I} D(x^{k+1}, x^k), x - x^{k+1} \rangle 
\]

\[
\geq -\delta_k \| x - x^{k+1} \| \quad \forall \ x \in K.
\]

This scheme includes an outer approximation of the operator \( Q \) in each iteration by set-valued operators \( Q^k \) and an inexact solution of the auxiliary problems. The family of monotone and continuous operators \( \{ L^k \} \) allows different types of approximations of the operator \( F \). The term \( \chi_k \nabla \mathcal{I} D(x^{k+1}, x^k) \) is made up by a positive parameter \( \chi_k \) and the gradient (with respect to the first vector argument) of a distance function \( D \).

As a special case we get the classical inexact PPA by setting \( Q = F + Q, F = 0, Q^k = Q, \forall k, \)

\( L^k = 0, \forall k, \) and \( D(x, y) = \frac{1}{2} \| x - y \|^2. \) A general inexact APP-scheme emerges from \( (P_{\delta}^k) \) by choosing \( F = F + Q, Q^k = 0, \forall k, \) and

\[
D(x, y) = h(x) - h(y) - \langle \nabla h(y), x - y \rangle \quad (1)
\]

with \( h \) continuously differentiable and \( \nabla h \) Lipschitz on \( K \). Then, \( L^k + \chi_k \nabla h \) plays the role of the auxiliary operator. Kaplan/Tichatschke showed in (5), that in scheme \( (P_{\delta}^k) \) it is possible to take a distance function like in (1) but with a Bregman-function \( h \), although the gradient map of a Bregman-function is not Lipschitz.
In our PAP-method with logarithmic-quadratic distances, $D$ is declared with the help of the following function which was first introduced by Auslender (1): For $v \in \mathbb{R}^p_{++}$ define
\[
d(u,v) := \begin{cases} 
\sum_{i=1}^p u_i^2 - u_i v_i - v_i^2 \log \frac{u_i}{v_i} & \text{if } u \in \mathbb{R}^p_{++} \setminus \{0\}, \\
+\infty & \text{otherwise}.
\end{cases}
\]
$d(\cdot, v)$ is a proper, lower semi-continuous and convex function, nonnegative and can be treated as unconstrained ones, because all iterates will automatically belong to the interior of the restriction set $K$.

Two properties of $D$ are important: First of all $\nabla I D(\cdot, x^k)$ is strictly monotone for all $x^k \in \text{int}(K)$. Together with a positive parameter $\chi_k$ this ensures that if the auxiliary problems $(P^k_\delta)$ are solvable they are uniquely solvable. This regularization effect enables us to deal with ill-posed problems. Second, it holds that the effective domain of $\nabla I D(\cdot, x^k)$ coincides with $\text{int}(K)$. This leads to an interior-point-effect, which means that the auxiliary problems $(P^k_\delta)$ can be treated as unconstrained ones, because all iterates will automatically belong to the interior of the restriction set $K$.

### 3. CONVERGENCE ANALYSIS

The assumptions in our convergence theorem are not stronger than those typically made for the PPA with Bregman-functions or the APP.

Apart from the already mentioned properties of the involved operators, we need that $\text{dom}(Q) \cap K$ is a nonempty and closed set and $\text{ri}(\text{dom}(Q)) \cap \text{int}(K) \neq \emptyset$. Further, the operators $F - L^k$ must fulfill a sort of Dunn-property and the family $\{L^k\}$ a continuity-property which is especially fulfilled if we have the uniformly Lipschitz continuity of the operators $L^k$. To approximate the operator $Q$ one can for example choose the $\epsilon$-enlargements $Q_{\epsilon_k}$ with $\epsilon_k \geq 0, \forall k$ and $\sum_{k=1}^{\infty} \epsilon_k < +\infty$.

The regularization parameter $\chi_k$ can vary from iteration to iteration, but has to be greater than a special positive constant. The error tolerance criterion is simply
\[
\sum_{k=0}^{\infty} \|\epsilon_{k+1}\| < +\infty
\]
which can easily be implemented.

If $VI(F, Q, K)$ is solvable we can prove convergence of the iterates $\{x^k\}$ generated by the LQPAP-method towards a solution.

### 4. CONCLUSIONS

We considered a general iteration scheme for solving variational inequalities, which can be viewed as an extension of the auxiliary-problem-principle. As regularization term we use a logarithmic-quadratic function that leads to an interior-point-effect. In contrast to the usage of Bregman distances we don’t have to require paramonotonicity of the operator of the variational inequality which opens our algorithm to a wider class of problems.

### REFERENCES


