LEVEL SET METHOD FOR SHAPE AND TOPOLOGY OPTIMIZATION OF CONTACT PROBLEMS

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1. INTRODUCTION

This paper deals with topology and shape optimization of an elastic contact problems. The shape optimization problem for elastic contact problem is formulated. Shape as well as topological derivatives formulae of the cost functional are provided using material derivative and asymptotic expansion methods, respectively. These derivatives are employed to formulate necessary optimality condition for simultaneous shape and topology optimization. Level set based numerical algorithm for the solution of the shape optimization problem is proposed. Numerical examples are provided and discussed.

2. PROBLEM FORMULATION

Consider deformations of an elastic body occupying two – dimensional domain Ω with the smooth boundary Γ. Assume $\Omega \subset D$ where D is a bounded smooth hold – all subset of R^2. The body is subject to body forces $f(x) = (f_1(x), f_2(x))$, $x \in \Omega$. Moreover, surface tractions $p(x) = (p_1(x), p_2(x))$, $x \in \Gamma$, are applied to a portion Γ_1 of the boundary Γ. We assume, that the body is clamped along the portion Γ_0 of the boundary Γ, and that the contact conditions are prescribed on the portion Γ_2, where $\Gamma_i \cap \Gamma_j = \emptyset$, $i \neq j$, $i, j = 0, 1, 2$, $\Gamma = \bar{\Gamma}_0 \cup \bar{\Gamma}_1 \cup \bar{\Gamma}_2$.

We denote by $u = (u_1, u_2)$, $u = u(x)$, $x \in \Omega$, the displacement of the body and by $\sigma(x) = \{\sigma_{ij}(u(x))\}$, $i, j = 1, 2$, the stress field in the body. Consider elastic bodies obeying Hooke's law, i.e., for $x \in \Omega$ and $i, j, k, l = 1, 2$

$$\sigma_{ij}(u(x)) = a_{ijkl}(x) e_{kl}(u(x)). \qquad (1)$$

We use here and throughout the paper the summation convention over repeated indices (2). The strain $e_{kl}(u(x))$, $k, l = 1, 2$, is defined by:

$$e_{kl}(u(x)) = \frac{1}{2}(u_{k,l}(x) + u_{l,k}(x)), \qquad (2)$$

where $u_{k,l}(x) = \frac{\partial u_k(x)}{\partial x_l}$. The stress field σ satisfies the system of equations (2)

$$-\sigma_{ij}(x)_{,j} = f_i(x) \quad x \in \Omega, i, j = 1, 2, \qquad (3)$$

where $\sigma_{ij}(x)_{,j} = \frac{\partial \sigma_{ij}(x)}{\partial x_j}$, $i, j = 1, 2$. The following boundary conditions are imposed

$$u_i(x) = 0 \quad \text{on} \quad \Gamma_0, \; i = 1, 2, \qquad (4)$$

$$\sigma_{ij}(x) n_j = p_i \quad \text{on} \quad \Gamma_1, \; i, j = 1, 2, \qquad (5)$$

$$u_N \leq 0, \; \sigma_N \leq 0, \; u_N \sigma_N = 0 \quad \text{on } \Gamma_2, \qquad (6)$$

$$| \sigma_T | \leq 1, \quad u_T \sigma_T + | u_T | = 0 \; \text{ on } \Gamma_2, \qquad (7)$$

where $n = (n_1, n_2)$ is the unit outward versor to the boundary Γ. Here $u_N = u_i n_i$ and $\sigma_N = \sigma_{ij} n_i n_j$, $i, j = 1, 2$, represent the normal components of displacement u and stress σ, respectively. The tangential components of displacement u and stress σ are given by $(u_T)_i = u_i - u_N n_i$ and $(\sigma_T)_i = \sigma_{ij} n_j - \sigma_N n_i$, $i, j = 1, 2$, respectively. $| u_T |$ denotes the Euclidean norm in R^2 of the tangent vector u_T. The results concerning the existence of solutions to (1) - (7) can be found in (2).

Let us recall from (3) the cost functional approximating the normal contact stress on the contact boundary

$$J_\phi(u(\Omega)) = \int_{\Gamma_2} \sigma_N(u) \phi_N(x) ds, \qquad (8)$$

depending on the auxiliary given bounded function $\phi(x)$, $x \in \Omega$. Let us denote by U_{ad} the set of admissible domains. Consider the following shape optimization problem: For a given function ϕ, find a domain $\Omega^\star \in U_{ad}$ such that

$$J_\phi(u(\Omega^\star)) = \min_{\Omega \in U_{ad}} J_\phi(u(\Omega)), \qquad (9)$$

where σ_N and ϕ_N are the normal components of the stress field σ corresponding to a solution u satisfying (1) - (7) and the function ϕ, respectively.

3. OPTIMALITY CONDITIONS

In the paper the optimality conditions for structural optimization problem (9) are formulated. Using material derivative method (1; 3) as well as asymptotic expansion method (1; 4) we calculate shape as well as topological derivatives of the cost functional (8). Finally the optimality condition for simultaneous shape and topology optimization problem is formulated.

4. LEVEL SET METHODS

In the paper the level set method (5) is employed to solve numerically problem (9). Consider the evolution of a domain Ω under a velocity field V. Let $t > 0$ denote the time variable. Under the mapping $T(t, V)$ we have

$$\Omega_t = T(t, V)(\Omega) = (I + tV)(\Omega), \quad t > 0.$$

By Ω_t^- we denote the interior of the domain Ω_t and by Ω_t^+ we denote the outside of the domain Ω_t. The domain Ω_t and its boundary $\partial\Omega_t$ are defined by a function $\Phi = \Phi(x, t) \; : \; R^2 \times [0, t_0) \to R$ satisfying

$$\begin{cases} \Phi(x,t) \; = \; 0, & \text{if } x \in \partial\Omega_t, \\ \Phi(x,t) \; < \; 0, & \text{if } x \in \Omega_t^-, \\ \Phi(x,t) \; > \; 0, & \text{if } x \in \Omega_t^+, \end{cases} \qquad (10)$$

i.e., the boundary $\partial\Omega_t$ is the level curve of the function Φ. Assume that velocity field V is known for every point x lying on the boundary $\partial\Omega_t$, i.e., with $\Phi(x, t) = 0$. Therefore the equation governing the evolution of the interface in $D \times [0, t_0]$ has the form (5)

$$\Phi_t(x, t) + V(x, t) \cdot \nabla_x \Phi(x, t) = 0, \qquad (11)$$

where Φ_t denotes a partial derivative of Φ with respect to the time variable t.

5. NUMERICAL METHODS

The structural optimization problem (9) is solved numerically as the shape and topology optimization problem. First the shape optimization problem is solved using the described level set method. In equation (11) velocity field V is set equal to the shape gradient of the cost functional (8). When the decrease in the cost functional value is less than the prescribed tolerance value the topology optimization problem is solved. The finite element method is used as the discretization method. Numerical results are provided. Obtained numerical results shows that the proposed algorithm allows for significant improvments from one iteration to the next.

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