REGULARIZED INCOMPLETE OBLIQUE PROJECTIONS METHOD FOR SOLVING LEAST-SQUARES PROBLEMS IN IMAGE RECONSTRUCTION

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1. INTRODUCTION

Large and sparse systems of linear equations arise in many important applications [1] as radiation therapy treatments planning, computational mechanics, optimization and in image processing problems like electromagnetic geotomography [2].

In practice, problems coming from the tomographic image reconstruction are in general inconsistent and of deficient rank. Those characteristics imply projection methods are particularly useful for solving them [1].

Many problems in the field of tomographic image reconstruction are modeled by the linear least-squares problem, that is : find $x^* \in \Re^n$ such that $\min_x \|Ax-b\|^2_{D_m}$, where A is an $m \times n$ matrix and $b \in \Re^m$, where $\|.\|_{D_m}$ denotes a weighted norm, and D_m is a positive definite matrix.

C. Popa has developed an extension of ART (Algebraic Reconstruction Technique) [1], called KERP [2], which converges for inconsistent systems, and more recently in [3] the authors showed its efficiency in the case of rank-deficient systems. Within the framework of the Projected Aggregation Methods (PAM) we have developed acceleration schemes based on projecting the search directions onto the aggregated hyperplanes, with excellent results in both consistent and inconsistent systems [4,5]. In particular, the IOP algorithm [5] uses a scheme of incomplete oblique projections onto the solution set of the augmented system $Ax - r = b$, which converges to a weighted least squares solution of the system $Ax = b$. It is known that not always the minimum norm solution turns out to be the closest to the true image. In this paper, as other authors like [3], we consider the regularized weighted least squares problem

$$\min_{x\in\Re^n} \frac{1}{2}\|Ax-b\|^2_{D_m} + \beta R(x) \qquad (1)$$

where D_m is a matrix of weights of data, and the second term is a function that penalizes the image roughness. The discrete smoothing norm in the previous problem can be defined (see [3]) as $R(x) = 2U(x)$, where

$$U(x) = \sum_{j=1}^{n}\sum_{i\in S_j} w_{ji} V(x_j - x_i, \delta),$$

S_j is a set of indices of the nearest neighborhood of pixel j, w_{ji} is a factor of weight, and $V(x_j - x_i, \delta)$ is a potential function. As it can be seen in the literature [3], there are several proposals aiming at the same objective. We have adopted $V(x_j - x_i, \delta) = (\frac{x_j - x_i}{\delta})^2$. This gives raise to the problem

$$\min_{x\in\Re^n} \frac{1}{2}\|Ax-b\|^2_{D_m} + \frac{1}{2}x^T M x, \qquad (2)$$

where the matrix M is positive definite, depending upon the weights w_{ji} and the S_j sets of indices of the nearest neighborhood of each pixel j.

As in [5], we define two convex sets in the $(2n+m)$- dimensional space $\Re^{2n+m}$, denoting by $[u;v]$ the vertical concatenation of $u \in \Re^n$, with $v \in \Re^{m+n}$,

$$\mathcal{P} = \{p : p = [x;r], x \in \Re^n, r \in \Re^{m+n}\} \quad (3)$$

being $r = [r_1; r_2] \in \Re^{m+n}$, such that satisfy $Ax - r_1 = b$, $M^{\frac{1}{2}} x - r_2 = 0$, and

$$\mathcal{Q} = \{q : q = [x; 0], x \in \Re^n, 0 \in \Re^{m+n}\}, \quad (4)$$

adopting the distance $d(p, q) = \|p - q\|_D$, for all $p \in \mathcal{P}$, $q \in \mathcal{Q}$. D is a diagonal matrix of order $2n + m$, whose n first elements are 1's, and the next m coincide with those of D_m, and the last n elements are 1's.

By means of a direct application of the Karush-Kuhn-Tucker (KKT) conditions [1] to the problem

$$\min\{\|p - q\|_D^2 : \forall p \in \mathcal{P}, \forall q \in \mathcal{Q}\} \quad (5)$$

it is possible to prove (see [5]) that this is equivalent to (2). This observation led us to use the IOP algorithm for solving (2), applying an alternate projections scheme between the sets $\mathcal{P}$ and $\mathcal{Q}$, similar to the original development in [5].

In the following sections we will present the RIOP algorithm based on the same scheme of the IOP algorithm, together with some related results needed for defining it and the corresponding convergence theory. In the last Section we will report numerical experiences for comparing the performance of the RIOP algorithm with the version of Kaczmarz Extended (KERP)[3] using simulated reconstruction problems in borebole electromagnetic geotomography.

2. INCOMPLETE OBLIQUE PROJECTION ALGORITHM

In order to solve the regularized weighted least squares problem (2) we consider its equivalence with (5). This observation led us to apply an alternate projections scheme between the sets $\mathcal{P}$ and $\mathcal{Q}$, but replacing the computation of the exact projections onto $\mathcal{P}$ by suitable incomplete or approximate projections, according to IOP algorithm in [5]. In order to compute the incomplete projections onto $\mathcal{P}$ we apply our ACCIM algorithm [4,5], which uses simultaneous projections onto the hyperplanes of the augmented system $Ax - r_1 = b_1$ and $Mx - \tilde{r}_2 = 0$, and is very efficient for solving consistent problems and convenient for computing approximate projections with some required properties, as explained in [5].

The diagonal matrix $D_{m+n} \in \Re^{m+n,m+n}$ is defined in such a way of allowing us to modify the weights of the residuals r_1 and $\tilde{r}_2$. The idea is to strongly penalize r_1, and to use the Euclidean norm in regard to $\tilde{r}_2$ for diminishing its influence in the general procedure.

3. NUMERICAL RESULTS AND CONCLUSIONS

In the full paper we will present a comparison of the results obtained with RIOP, IOP [5],KERP [2], RKERP [3], for several image reconstruction problems. As it will be seen the RIOP efectiveness is remarkable in several problems. In forthcoming papers we will analyze alternative functions for penalizing the least squares problem, aiming at smoothing the image. Likewise, in regard to the same $R(x)$ used in this paper we will study the effect of adding neighboring pixels using a wider radius, and also what suitable weights are from a practical viewpoint.

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