NONLINEAR BOUNDARY CONTROL FOR A CLASS OF 1-D NONLINEAR PARABOLIC PDEs

Miroslav Krstic and Rafael Vazquez
University of California, San Diego, USA, krstic@ucsd.edu
Keywords: boundary control, backstepping, parabolic PDEs

Abstract. Certain classes of parabolic PDEs that arise in chemical process control include nonlinear Volterra series operators on the right hand sides of the PDEs, with the spatial coordinate as the integration variable. A stabilizing feedback law in the form of a Volterra series will be presented. The kernels of the series are given through a sequence of hyperbolic PDEs on spatial domains of increasing dimension and dependent on the Volterra kernels of the plant nonlinearity. We present a priori estimates for the control kernels, the convergence analysis of the Volterra nonlinear feedback operator, and numerical results for several benchmark nonlinear PDEs.

1. Introduction

Boundary control of linear parabolic PDEs is a well established subject with extensive literature. On the other hand, boundary control of nonlinear parabolic PDEs is still an open problem as far as general classes of systems are concerned.

Our method is a direct infinite dimensional extension of the finite-dimensional feedback linearization/backstepping approaches and employs spatial Volterra series nonlinear operators. We only sketch our method here; a two-part paper [3] has been submitted presenting the method and its properties in full detail, with examples. This result solves open problem 5.1 in the Unsolved Problems volume [1].

2. Volterra Series

Volterra series represent general solutions for nonlinear equations and are widely studied in the literature [2]. A (spatial) Volterra series is defined as

\[ F[u] = \sum_{n=1}^{\infty} \int_{0}^{x} \int_{0}^{\xi_1} \cdots \int_{0}^{\xi_{n-1}} f_n(x, \xi_1, \ldots, \xi_n) \left( \prod_{j=1}^{n} u(t, \xi_j) \right) \times d\xi_1 \cdots d\xi_n, \]

where \( f_n \) is known as the \( n \)-th (triangular) kernel of \( F \).

3. Outline of the Method

We consider the stabilization problem for the plant

\[ \begin{align*}
    u_t &= u_{xx} + \lambda(x)u + F[u] + uH[u], \quad (2) \\
    u_x(0, t) &= qu(0, t) \quad (3) \\
    u(1, t) &= U(t), \quad (4)
\end{align*} \]

where \( F[u] \) and \( H[u] \) are Volterra series and \( U(t) \) the actuation variable. In [3] we show how nonlinear plants found in applications can be written in the form (2)–(4).

We solve the problem by mapping \( u \) into a target system \( w \) which verifies

\[ \begin{align*}
    w_t &= w_{xx} - cw, \quad (5) \\
    w_x(0, t) &= \bar{q}w(0, t) \quad (6) \\
    w(1, t) &= 0,
\end{align*} \]

where \( \bar{q} = \max\{0, q\} \). For mapping \( u \) into \( w \) we use a Volterra transformation

\[ w = u - K[u]. \quad (7) \]

In [3] we derive the equations that the kernels \( k_n \) of \( K \) in (7) verify. It is a set of linear hyperbolic PDEs. For each \( k_n \), we get a PDE evolving on a domain of dimension \( n + 1 \) and...
with a domain shape in the form of a “hyper-pyramid,” \( 0 \leq \xi_n \leq \xi_{n-1} \ldots \leq \xi_1 \leq x \leq 1 \).

The equations can be solved recursively, i.e., first for \( k_1 \) (which verifies an autonomous equation), then for \( k_2 \) (which is coupled with \( k_1 \)) using the solution for \( k_1 \), and so on. We also show in [3] that the Volterra series defined by the \( k_n \)'s in (7) is always convergent and invertible (at least locally).

Once we have the \( k_n \)'s, the stabilizing control law is determined by (7) at \( x = 1 \)

\[
U(t) = \sum_{n=1}^{\infty} \int_{0}^{1} \int_{0}^{\xi_1} \ldots \int_{0}^{\xi_{n-1}} k_n(1, \xi_1, \ldots, \xi_n) \left( \prod_{j=1}^{n} u(t, \xi_j) \right) \\
\times d\xi_1 \ldots d\xi_n. \tag{8}
\]

In [3], using the invertibility properties of \( K \) and the exponential stability of (5)–(6), we show that the origin of the closed-loop system (2)–(4) with control law (8) is exponentially stable in the \( L^2 \) and \( H^1 \) norms (at least locally). We also illustrate this result with numerical simulations of several examples of interest.

REFERENCES

